ALGEBRAICAL PROBLEMS.

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ALGEBRAICAL PROBLEMS,

PRODUCING

SIMPLE AND QUADRATIC EQUATIONS,

WITH

THEIR SOLUTIONS;

DESIGNED AS

AN INTRODUCTION TO THE HIGHER BRANCHES OF ANALYTICS:

TO WHICH IS ADDED,

AN APPENDIX,

CONTAINING A COLLECTION OF PROBLEMS ON THE NATURE AND SOLUTION OF EQUATIONS OF HIGHER DIMENSIONS.

BY

1786 – 1868

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ADVERTISEMENT.

The following pages, of which *eight* large editions have been favourably received by the public, contain a collection of Algebraical Problems, designed to point out the various methods employed by Analysts in the Solution of Equations. They were originally drawn up for the use of the younger Students of St. John's College, in performance of the duties attached to the office of Sadlerian Algebra Lecturer; and were printed with the approbation of the late Very Rev. Dr. Wood, who had himself at one time designed a similar publication, but from more important occupations had not found leisure to collect materials for the work. The Examples are arranged in the usual manner: 1. Simple Equations; 2. Pure Quadratics, and others which may be solved without completing the square; and 3. Adfected Quadratics. Utility being the sole object of this Publication, wherever a proper Example occurred, it has been taken without hesitation, or altered to suit the purpose. Many have been selected from the questions which for a very long period have been proposed annually to the Freshmen in the College Examinations at St. John's: and several successive Editions of the Work have been benefited by

the contributions of friends. At the head of each Section are given the common Rules; and the whole concludes with a Collection of Problems without Solutions, for the Exercise of the Learner.

To the Sixth Edition was added an Appendix, containing a Collection of Problems in Arithmetical, Geometrical, and Harmonical Progressions; and another on the nature of Equations, and the solution of those of higher dimensions.

And the Ninth Edition has been increased by an additional Section on the Solution of Indeterminate Equations and Problems, with a corresponding portion of the Praxis.

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ALGEBRAICAL PROBLEMS.

SECTION I.

DEFINITIONS.

(1.) An Equation is a proposition, which declares the equality of two quantities, expressed algebraically.

This is done by connecting these quantities by the sign $(=)$; thus, $x - 4 = 6 - x$ is an equation expressing the equality of the quantities $x - 4$ and $6 - x$, i. e. that a certain unknown quantity x is so connected with two quantities 4 and 6, that it exceeds the one (4) by as much as it falls short of the other (6). Also $x - 5 = 0$ is an equation which asserts that $x - 5$ is equal to nothing, and therefore that the positive part of the expression is equal to the negative part.

(2.) A Simple Equation is one which, when cleared of fractions and surds, contains only the first power of the unknown quantity.

(3.) A Quadratic Equation, or an equation of two dimensions, is one into which the square of the unknown quantity enters, with or without the simple power.

(4.) A Cubic Equation, or an equation of three dimensions, is one into which the cube of the unknown quantity enters, with or without the simple and quadratic powers.

(5.) In general, the index of the highest power of the unknown quantity denotes the number of dimensions of the equation.

(6.) A Pure Quadratic is one into which only the square of the unknown quantity enters.

(7.) An Adfected Quadratic is one which involves the square of the unknown quantity, and also the simple power and known quantities.

Thus, $ax^2 + b = 0$ is a pure quadratic,

and $ax^2 + bx + c = 0$ is an adfected quadratic.

(8.) The Resolution of Equations is the determining, from some quantities given, the values of others which are unknown, so that these latter may answer certain conditions proposed.

(9.) And these values are called Roots of the Equation.

(10.) Known quantities are usually expressed by the first letters of the alphabet, a, b, c, &c.; and unknown quantities by the last, v, x, y, &c. And this must be always understood, unless the contrary be expressed.

AXIOMS.

(11.) If equal quantities be added to equal quantities, the sums will be equal.

(12.) If equal quantities be taken from equal quantities, the remainders will be equal.

(13.) If equal quantities be multiplied by the same or equal quantities, the products will be equal.

(14.) If equal quantities be divided by the same or equal quantities, the quotients will be equal.

(15.) If the same quantity be added to and subtracted from another, the value of the latter will not be altered.

(16.) If a quantity be both multiplied and divided by another, its value will not be altered.

(17.) Any quantity may be transposed from one side of an equation to the other, by changing its sign:

Because, in this transposition, the same quantity is merely subtracted from each side of the equation; and (12) if equals be taken from equals, the remainders are equal.

Thus, if $x + 9 = 15$, and 9 be subtracted from each side,

$x = 15 - 9$, or 6. Also, if $x + b = a$, and b be subtracted from each side, $x = a - b$. And if $x - c = d$, and c be added to each side, $x = d + c$.

Also, if $5x - 7 = 2x + 2$, and $2x$ be taken from each side, $5x - 2x - 7 = 2$, or $3x - 7 = 2$; and if -7 be subtracted, or (which is the same thing) if $+7$ be added to each side $3x = 2 + 7 = 9$.

Also, if $x - a + b = c - 3x$, then, by subtracting $-a + b - 3x$ from each side, we have $x + 3x = a - b + c$.

COR. 1. Hence, if the signs of all the terms on each side of an equation be changed, the two sides still remain equal; because in this change every term is transposed.

COR. 2. Hence, when the known and unknown quantities are connected in an equation by the signs + or −, they may be separated by transposing the known quantities to one side, and the unknown to the other.

COR. 3. Hence also, if any quantity be found on both sides of an equation, it may be taken away from each; thus, if $x + y = 5 + y$, then $x = 5$. If $x - b = c + d - b$, then $x = c + d$.

(18.) If every term on each side of an equation be multiplied by the same quantity, the results will be equal:

Because in multiplying every term on each side by any quantity, the value of the whole side is multiplied by that quantity; and (13) if equals be multiplied by the same quantity, the products will be equal.

Thus, if $x = 5 + a$, then $6x = 30 + 6a$, by multiplying every term by 6.

COR. 1. Hence an equation, of which any part is fractional, may be reduced to an equation expressed in integers, by multiplying every term by the denominator of the fraction. If there be more fractions than one in the given equation, it may be so reduced by multiplying every term either by the product of the denominators, or by a common multiple of them; and if the least common multiple be used, the equation will be in its lowest terms.

Thus, if $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13$; if every term be multiplied by 12, which is the least common multiple of 2, 3, 4; $6x+4x+3x=156$.

COR. 2. Hence also, if every term on both sides have a common multiplier or divisor, that common multiplier or divisor may be taken away;

Thus, if $ax^2+abx=cdx$; each term being divided by the common multiplier, x, $ax+ab=cd$.

Also, if $\frac{5x}{4}+\frac{x+6}{4}=\frac{4x+7}{4}$, then also $5x+x+6=4x+7$;

Also, if $\frac{ax+ab}{c}=\frac{ad}{c}+\frac{4ax}{c}$, then, multiplying by $\frac{c}{a}$, $x+b=d+4x$.

Also, if $(a^2+x^2)^{\frac{3}{2}}=3x^2 \,.\, (a^2+x^2)^{\frac{1}{2}}$, then dividing by $(a^2+x^2)^{\frac{1}{2}}$, $a^2+x^2=3x^2$.

COR. 3. Also, if each member of the equation have a common divisor, the equation may be reduced by dividing both sides by that common divisor;

Thus, if $ax^2-a^2x=abx-a^2b$, each side is divisible by $ax-a^2$, whence $x=b$.

COR. 4. Hence also any term of an equation may be made a square, by multiplying all the terms of the equation by the quantities necessary; as, if $ax^2+bcx=cd^2$, the first term may be made a square by multiplying each term by a, and $a^2x^2+abcx=acd^2$.

(19.) If each side of an equation be raised to the same power, the results are equal;

Thus, if $x=6$, $x^2=36$; if $x+a=y-b$, then $x^2+2ax+a^2=y^2-2by+b^2$;

And if the same roots be extracted on each side, the results are equal:

Thus, if $x^2 = 49$, $x = 7$; if $x^2 = a^2b^2$, then $x = ab$; if $x^2 + 2x + 1 = y^2 - y + \frac{1}{4}$, then $x + 1 = y - \frac{1}{2}$, and if $x^2 - 4ax + 4a^2 = y^2 + 6by + 9b^2$, then $x - 2a = y + 3b$.

For (13 and 14) when equal quantities on each side of an equation are multiplied or divided by equal quantities, the results will be equal.

Cor. Hence, if that side of the equation which contains the unknown quantity be a perfect square, cube, or other power, by extracting the square root, cube root, &c. of both sides, the equation will be reduced to one of lower dimensions:

Thus, if $x^2 + 8x + 16 = 36$, then $x + 4 = 6$,
if $x^3 + 3x^2 + 3x + 1 = 27$, then $x + 1 = 3$,
if $x^4 + 2x^3 + x^2 = 100$, then $x^2 + x = 10$.

(20.) Any equation may be cleared of a single radical quantity by transposing all the other terms to the contrary side, and raising each side to the power denominated by the surd. If there are more than one surd, the operation must be repeated.

Thus, if $x = \sqrt{ax + b^2}$, by squaring each side $x^2 = ax + b^2$, which is free from surds.

Also, if $\sqrt{x^2 + 7} + x = 7$,

then (17) by transposition, $\sqrt{x^2 + 7} = 7 - x$;

and (19) by squaring each side, $x^2 + 7 = 49 - 14x + x^2$, which is free from surds.

Also, if $x + \sqrt[3]{a^2x} = b$,

then (17) by transposition, $\sqrt[3]{a^2x} = b - x$;

and (19) by cubing each side, $a^2x = b^3 - 3b^2x + 3bx^2 - x^3$, which is free from surds.

$$\text{Also, if } \sqrt{x^2 + \sqrt{x^2 + 21}} - 1 = x,$$

$$\text{then (17) by transposition, } \sqrt{x^2 + \sqrt{x^2 + 21}} = x + 1,$$

$$\text{and (19) by squaring each side, } x^2 + \sqrt{x^2 + 21} = x^2 + 2x + 1;$$

$$\therefore \text{ (17. Cor. 3.) } \sqrt{x^2 + 21} = 2x + 1,$$

and (19) by squaring each side, $x^2 + 21 = 4x^2 + 4x + 1$, which is free from surds.

And, if $\sqrt[3]{a^2x + \sqrt{a^3x^3}} = c$,

(19) by cubing each side, $a^2x + \sqrt{a^3x^3} = c^3$,

and (17) by transposition, $\sqrt{a^3x^3} = c^3 - a^2x$;

$\therefore$ (19) by squaring each side, $a^3x^3 = c^6 - 2a^2c^3x + a^4x^2$, which is free from surds.

(21.) Any proportion may be converted into an equation; for the product of the extremes is equal to the product of the means.

Let $a : b :: c : d$, by the nature of proportion $\frac{a}{b} = \frac{c}{d}$;

$\therefore$ (18. Cor. 1.) $ad = bc$.

(22.) Examples *in which the preceding Rules are applied, in the Solution of Equations.*

1. Given $4x + 36 = 5x + 34$, to find the value of x.

(17) By transposition, $36 - 34 = 5x - 4x$,
and $\therefore$ $2 = x$*.

2. Given $x - 7 = \frac{x}{5} + \frac{x}{3}$, to find the value of x.

Here 15, the product of 3 and 5, being their least common multiple, every term must be multiplied by it (18. Cor. 1.), and $15x - 105 = 3x + 5x$;

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* A Simple Equation can have only one solution, or there can be only one value of the unknown quantity which will satisfy that equation.

For every simple equation may be reduced to the form $ax + b = 0$. If possible $\therefore$ let there be two values of x, α and β, which satisfy this equation.

Then $a\alpha + b = 0$
and $a\beta + b = 0$,

$\therefore$ by subtraction, $a \, . \, (\alpha - \beta) = 0$.

But a cannot be $= 0$; inasmuch as the proposed equation would not be an equation with respect to x,

$\therefore$ $\alpha - \beta = 0$, or $\alpha = \beta$,

i. e. α and β cannot be different values; or there is only one value of x which satisfies the conditions of the equation.

$\therefore$ (17) by transposition, $15x - 3x - 5x = 105$,

or $7x = 105$;

and $\therefore$ (18. Cor. 2.) $x = \dfrac{105}{7} = 15$.

3. Given $3ax - 4ab = 2ax - 6ac$, to find the value of x in terms of b and c.

(18. Cor. 2.) dividing every term by a, $3x - 4b = 2x - 6c$;

$\therefore$ (17) by transposition, $3x - 2x = 4b - 6c$,

or $x = 4b - 6c$.

4. Given $3x^2 - 10x = 8x + x^2$, to find the value of x.

(18. Cor. 2.) dividing every term by x, $3x - 10 = 8 + x$;

$\therefore$ by transposition, $3x - x = 8 + 10$,

or $2x = 18$;

$\therefore$ (18. Cor. 2.) $x = \dfrac{18}{2} = 9$.

5. Given $\dfrac{x}{2} + \dfrac{x}{3} = \dfrac{x}{4} + 7$, to find the value of x.

Here 12 is the least common multiple of 2, 3, and 4; (18. Cor. 1.) multiplying both sides of the equation therefore by 12, $6x + 4x = 3x + 84$;

$\therefore$ (17) by transposition, $6x + 4x - 3x = 84$,

or $7x = 84$;

$\therefore$ (18. Cor. 2.) $x = \dfrac{84}{7} = 12$.

6. Given $\dfrac{x-5}{4} + 6x = \dfrac{284 - x}{5}$, to find the value of x.

(18. Cor. 1.) multiplying by 20, the least common multiple of 4 and 5,

$$5x - 25 + 120x = 1136 - 4x;$$

$\therefore$ (17) by transposition, $5x + 120x + 4x = 1136 + 25$,

or $129x = 1161$;

$\therefore$ (18. Cor. 2.) $x = \dfrac{1161}{129} = 9$.

7. Given $x + \frac{11 - x}{3} = \frac{19 - x}{2}$, to find the value of x.

(18. Cor. 1.) multiplying by 6, the least common multiple of 2 and 3.

$$6x + 22 - 2x = 57 - 3x;$$

$$\therefore \text{ (17) by transposition, } 6x - 2x + 3x = 57 - 22,$$

$$\text{or } 7x = 35;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{35}{7} = 5.$$

8. Given $3x + \frac{2x + 6}{5} = 5 + \frac{11x - 37}{2}$, to find the value of x.

(18. Cor. 1.) multiplying by 10, the least common multiple of 2 and 5,

$$30x + 4x + 12 = 50 + 55x - 185;$$

$$\therefore \text{ (17) by transposition, } 12 - 50 + 185 = 55x - 30x - 4x,$$

$$\text{or } 147 = 21x;$$

$$\therefore \text{ (18. Cor. 2.) } \frac{147}{21} = 7 = x.$$

9. Given $\frac{6x - 4}{3} - 2 = \frac{18 - 4x}{3} + x$, to find the value of x.

(18. Cor. 1.) multiplying every term by 3,

$$6x - 4 - 6 = 18 - 4x + 3x;$$

$$\text{and } \therefore \text{ (17) by transposition, } 6x + 4x - 3x = 18 + 6 + 4,$$

$$\text{or } 7x = 28;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{28}{7} = 4.$$

10. Given $21 + \frac{3x - 11}{16} = \frac{5x - 5}{8} + \frac{97 - 7x}{2}$, to find the value of x.

Since 16 contains 8 and 2, a certain number of times exactly, it will be the least common multiple of 16, 8, and 2; and therefore (18. Cor. 1.) multiplying both sides of the equation by 16,

$$336 + 3x - 11 = 10x - 10 + 776 - 56x;$$

$\therefore$ (17) by transposition, $3x - 10x + 56x = 11 - 10 + 776 - 336$,

$$\text{or } 49x = 441;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{441}{49} = 9.$$

11. Given $x + \frac{3x - 5}{2} = 12 - \frac{2x - 4}{3}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by 6, the product of 2 and 3,

$$6x + 9x - 15 = 72 - 4x + 8;$$

$$\therefore \text{ (17) by transposition, } 6x + 9x + 4x = 72 + 8 + 15,$$

$$\text{or } 19x = 95;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{95}{19} = 5.$$

12. Given $3x - \frac{x - 4}{4} - 4 = \frac{5x + 14}{3} - \frac{1}{12}$, to find the value of x.

Since 12 is a multiple of 3 and 4, it is the least common multiple of 3, 4, and 12; therefore (18. Cor. 1.) multiplying both sides of the equation by 12,

$$36x - 3x + 12 - 48 = 20x + 56 - 1;$$

$\therefore$ (17) by transposition, $36x - 3x - 20x = 56 + 48 - 1 - 12$,

$$\text{or } 13x = 91;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{91}{13} = 7.$$

13. Given $\frac{x - 1}{7} + \frac{23 - x}{5} = 7 - \frac{4 + x}{4}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $4 \times 5 \times 7 = 140$,

$$20x - 20 + 644 - 28x = 980 - 140 - 35x;$$

$\therefore$ (17) by trans^{n}, $20x - 28x + 35x = 980 - 140 + 20 - 644$,

$$\text{or } 27x = 216;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{216}{27} = 8.$$

14. Given $\frac{7x + 5}{3} - \frac{16 + 4x}{5} + 6 = \frac{3x + 9}{2}$, to find the value of x.

(18. Cor. 1.) multiplying both sides by $2 \times 3 \times 5 = 30$,

$$70x + 50 - 96 - 24x + 180 = 45x + 135;$$

$\therefore$ (17) by transposition, $70x - 24x - 45x = 135 + 96 - 50 - 180$,

$$\text{or } x = 1.$$

15. Given $\frac{3x + 4}{5} - \frac{7x - 3}{2} = \frac{x - 16}{4}$, to find the value of x.

(18. Cor. 1.) multiplying by 20, the least common multiple of 2, 4, and 5,

$$12x + 16 - 70x + 30 = 5x - 80;$$

$\therefore$ (17) by transposition, $16 + 30 + 80 = 5x + 70x - 12x$,

$$\text{or } 126 = 63x;$$

$$\therefore \text{ (18. Cor. 2.) } \frac{126}{63} = 2 = x.$$

16. Given $\frac{17 - 3x}{5} - \frac{4x + 2}{3} = 5 - 6x + \frac{7x + 14}{3}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $3 \times 5 = 15$,

$$51 - 9x - 20x - 10 = 75 - 90x + 35x + 70;$$

$\therefore$ (17) by trans^n, $90x - 35x - 20x - 9x = 75 + 70 + 10 - 51$,

$$\text{or } 26x = 104;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{104}{26} = 4.$$

17. Given $x - \frac{3x - 3}{5} + 4 = \frac{20 - x}{2} - \frac{6x - 8}{7} + \frac{4x - 4}{5}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $2 \times 5 \times 7 = 70$,

$$70x - 42x + 42 + 280 = 700 - 35x - 60x + 80 + 56x - 56;$$

$\therefore$ (17) by transposition,

$$70x + 35x + 60x - 42x - 56x = 700 + 80 - 56 - 280 - 42,$$

or $67x = 402$;

$$\therefore \text{ (18. Cor. 2.) } x = \frac{402}{67} = 6.$$

18. Given $\frac{4x-21}{9} + 3\frac{3}{4} + \frac{57-3x}{4} = 241 - \frac{5x-96}{12} - 11x$, to find the value of x.

(18. Cor. 1.) multiplying by 36, the least common multiple of 4, 9, and 12,

$$16x - 84 + 135 + 513 - 27x = 8676 - 15x + 288 - 396x;$$

$\therefore$ (17) by transposition,

$$16x + 15x + 396x - 27x = 8676 + 288 + 84 - 135 - 513,$$

or $400x = 8400$;

$$\therefore \text{ (18. Cor. 2.) } x = \frac{8400}{400} = 21.$$

19. Given $\frac{6x+18}{13} - 4\frac{1}{6} - \frac{11-3x}{36} = 5x - 48 - \frac{13-x}{12} - \frac{21-2x}{18}$, to find the value of x.

(18. Cor. 1.) multiplying by 36×13, the least common multiple of the denominators,

$$216x + 648 - 2262 - 143 + 39x$$

$$= 2340x - 22464 - 507 + 39x - 546 + 52x;$$

$\therefore$ (17) by transposition, $648 + 22464 + 507 + 546 - 2262 - 143 = 2340x + 39x + 52x - 216x - 39x$,

or $21760 = 2176x$;

$$\therefore \text{ (18. Cor. 2.) } \frac{21760}{2176} = 10 = x.$$

20. Given $ax - \frac{a^2 - 3bx}{a} - ab^2 = bx + \frac{6bx - 5a^2}{2a} - \frac{bx + 4a}{4}$, to find the value of x.

(18. Cor. 1.) multiplying by $4a$, the least common multiple of the denominators,

$$4a^2x - 4a^2 + 12bx - 4a^2b^2 = 4abx + 12bx - 10a^2 - abx - 4a^2;$$

$$\therefore \text{ (17. Cor. 3.) } 4a^2x - 4a^2b^2 = 3abx - 10a^2;$$

$$\text{by transposition, } (4a^2 - 3ab) \,.\, x = 4a^2b^2 - 10a^2;$$

$$\text{(18. Cor. 2.) } (4a - 3b) \,.\, x = 4ab^2 - 10a;$$

$$\therefore\ x = \frac{4ab^2 - 10a}{4a - 3b}.$$

21. Given $\frac{7x + 16}{21} - \frac{x + 8}{4x - 11} = \frac{x}{3}$, to find the value of x.

Multiplying both sides of the equation by 21,

$$7x + 16 - \frac{21x + 168}{4x - 11} = 7x;$$

$$\therefore \text{ (17. Cor. 3.) } 16 = \frac{21x + 168}{4x - 11};$$

$$\therefore \text{ (18. Cor. 1.) } 64x - 176 = 21x + 168;$$

$$\therefore \text{ (17) by transposition, } 64x - 21x = 168 + 176,$$

$$\text{or } 43x = 344;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{344}{43} = 8.$$

22. Given $\frac{6x + 7}{9} + \frac{7x - 13}{6x + 3} = \frac{2x + 4}{3}$, to find the value of x.

Multiplying both sides of the equation by 9,

$$6x + 7 + \frac{21x - 39}{2x + 1} = 6x + 12;$$

$$\therefore \text{ (17. Cor. 3.) } \frac{21x - 39}{2x + 1} = 5;$$

$$\therefore \text{ (18. Cor. 1.) } 21x - 39 = 10x + 5;$$

$$\therefore \text{ (17) by transposition, } 21x - 10x = 39 + 5,$$

$$\text{or } 11x = 44;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{44}{11} = 4.$$

23. Given $\frac{4x + 3}{9} + \frac{7x - 29}{5x - 12} = \frac{8x + 19}{18}$, to find the value of x.

Multiplying both sides of the equation by 18,

$$8x + 6 + \frac{126x - 522}{5x - 12} = 8x + 19;$$

$$\therefore \text{ (17. Cor. 3.) } \frac{126x - 522}{5x - 12} = 13,$$

$$\text{and (18. Cor. 1.) } 126x - 522 = 65x - 156;$$

$$\text{(17) by transposition, } 61x = 366;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{366}{61} = 6.$$

24. Given $12 - x : \frac{x}{2} :: 4 : 1$, to find the value of x.

(21) Since the product of the extremes is equal to the product of the means,

$$12 - x = 4 \times \frac{x}{2} = 2x;$$

$$\therefore \text{ (17) by transposition, } 12 = 2x + x = 3x,$$

$$\text{and (18. Cor. 2.) } \frac{12}{3} = 4 = x.$$

25. Given $\frac{5x + 4}{2} : \frac{18 - x}{4} :: 7 : 4$, to find the value of x.

$$(21)\ \frac{5x+4}{2}\times 4=\frac{18-x}{4}\times 7,$$

$$\text{or } 10x+8=\frac{126-7x}{4};$$

$$\therefore\ (18.\ \text{Cor. } 2.)\ 40x+32=126-7x;$$

$$\therefore\ (17)\ \text{by transposition, } 40x+7x=126-32,$$

$$\text{or } 47x=94;$$

$$\therefore\ x=\frac{94}{47}=2.$$

26. Given $\sqrt{(4x+16)}=12$, to find the value of x.

(19) squaring both sides of the equation, $4x+16=144$;

$\therefore$ (17) by transposition, $4x=144-16=128$;

$$\therefore\ (18.\ \text{Cor. } 2.)\ x=\frac{128}{4}=32.$$

27. Given $\sqrt[3]{(2x+3)}+4=7$, to find the value of x.

(17) by transposition, $\sqrt[3]{(2x+3)}=7-4=3$;

$\therefore$ (19) cubing both sides of the equation, $2x+3=27$;

$\therefore$ (17) by transposition, $2x=27-3=24$,

$$\text{and } (18.\ \text{Cor. } 2.)\ x=\frac{24}{2}=12.$$

28. Given $\sqrt{(12+x)}=2+\sqrt{x}$, to find the value of x.

(19) squaring both sides of the equation,

$$12+x=4+4\sqrt{x}+x;$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 8=4\sqrt{x};$$

$$\text{and } (18.\ \text{Cor. } 2.)\ 2=\sqrt{x};$$

$$\therefore\ (19)\ 4=x.$$

29. Given $\sqrt{(x+40)}=10-\sqrt{x}$, to find the value of x.

(19) squaring both sides of the equation,

$$x + 40 = 100 - 20\sqrt{x} + x;$$

$$\therefore \text{ (17. Cor. 3.) } 20\sqrt{x} = 100 - 40 = 60,$$

$$\text{and (18. Cor. 2.) } \sqrt{x} = 3;$$

$$\therefore \text{ (19) } x = 9.$$

30. Given $\sqrt{(x - 16)} = 8 - \sqrt{x}$, to find the value of x.
(19) squaring both sides of the equation,

$$x - 16 = 64 - 16\sqrt{x} + x;$$

$$\therefore \text{ (17. Cor. 3.) } 16\sqrt{x} = 64 + 16 = 80;$$

$$\therefore \text{ (18. Cor. 2.) } \sqrt{x} = 5,$$

$$\text{and (19) } x = 25.$$

31. Given $\sqrt{(x - 24)} = \sqrt{x} - 2$, to find the value of x.
(19) squaring both sides of the equation,

$$x - 24 = x - 4\sqrt{x} + 4;$$

$$\therefore \text{ (17. Cor. 3.) } 4\sqrt{x} = 24 + 4 = 28;$$

$$\therefore \text{ (18. Cor. 2.) } \sqrt{x} = 7,$$

$$\text{and (19) } x = 49.$$

32. Given $\sqrt{(x - a)} = \sqrt{x} - \frac{1}{2}\sqrt{a}$, to find the value of x.

(19) squaring both sides of the equation,

$$x - a = x - \sqrt{ax} + \tfrac{1}{4}a;$$

$$\therefore \text{ (17. Cor. 3.) } \sqrt{ax} = a + \tfrac{1}{4}a = \frac{5a}{4};$$

$$\text{(19) } ax = \frac{25a^2}{16};$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{25a}{16}.$$

33. Given $\sqrt{5} \times \sqrt{(x + 2)} = \sqrt{5x} + 2$, to find the value of x.

(19) squaring both sides of the equation,

$$5x + 10 = 5x + 4\sqrt{5x} + 4;$$

$$\therefore \text{ (17. Cor. 3.) } 10 - 4 = 6 = 4\sqrt{5x};$$

$$\therefore \text{ (18. Cor. 2.) } \frac{6}{4} = \frac{3}{2} = \sqrt{5x},$$

$$\text{and (19) } \frac{9}{4} = 5x;$$

$$\therefore \text{ (18. Cor. 2.) } \frac{9}{20} = x.$$

34. Given $\sqrt{(4a + x)} = 2\sqrt{(b + x)} - \sqrt{x}$, to find the value of x.

(19) squaring both sides of the equation,

$$4a + x = 4 \,.\, (b + x) - 4\sqrt{(bx + x^2)} + x;$$

$$\text{(17. Cor. 3.) } 4a + 4\sqrt{(bx + x^2)} = 4 \,.\, (b + x);$$

$$\text{(18. Cor. 2.) } a + \sqrt{(bx + x^2)} = b + x;$$

$$\text{(17) by transposition, } \sqrt{(bx + x^2)} = b - a + x;$$

$$\text{(19) } bx + x^2 = (b - a)^2 + 2 \,.\, (b - a) \,.\, x + x^2;$$

$$\text{(17. Cor. 3.) } (2a - b) \,.\, x = (b - a)^2;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{(b - a)^2}{2a - b}.$$

35. Given $x + a + \sqrt{(2ax + x^2)} = b$, to find the value of x.

(17) by transposition, $\sqrt{(2ax + x^2)} = b - a - x$;

and (19) squaring both sides,

$$2ax + x^2 = (b - a)^2 - 2 \,.\, (b - a) \,.\, x + x^2;$$

$$\therefore \text{ (17. Cor. 3.) } 2ax + 2 \,.\, (b - a) \,.\, x = (b - a)^2,$$

$$\text{or } 2bx = (b - a)^2;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{(b - a)^2}{2b}.$$

36. Given $\frac{x - ax}{\sqrt{x}} = \frac{\sqrt{x}}{x}$ to find the value of x.

Multiplying both sides of the equation by $\sqrt{x}$,

$$x - ax = \frac{x}{x} = 1,$$

$$(1 - a) \,.\, x = 1,$$

and (18. Cor. 2.) $x = \frac{1}{1-a}$.

37. Given $\frac{\sqrt{x} + 28}{\sqrt{x} + 4} = \frac{\sqrt{x} + 38}{\sqrt{x} + 6}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $(\sqrt{x} + 4) \times (\sqrt{x} + 6)$,

$$x + 34\sqrt{x} + 168 = x + 42\sqrt{x} + 152;$$

$\therefore$ (17. Cor. 3.) taking $(x + 34\sqrt{x} + 152)$ from each side of the equation,

$$16 = 8\sqrt{x};$$

$\therefore$ (18. Cor. 2.) $2 = \sqrt{x}$;

$\therefore$ (19) $4 = x$.

38. Given $\frac{\sqrt{x} + 2a}{\sqrt{x} + b} = \frac{\sqrt{x} + 4a}{\sqrt{x} + 3b}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $(\sqrt{x} + b) \,.\, (\sqrt{x} + 3b)$,

$$x + (2a + 3b) \,.\, \sqrt{x} + 6ab = x + (4a + b) \,.\, \sqrt{x} + 4ab;$$

$\therefore$ (17. Cor. 3.) $(2a - 2b) \,.\, \sqrt{x} = 2ab$,

or (18. Cor. 2.) $\sqrt{x} = \frac{ab}{a-b}$;

$\therefore$ (19) $x = \left(\frac{ab}{a-b}\right)^2$.

39. Given $\frac{\sqrt{ax}-b}{\sqrt{ax}+b}=\frac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $(\sqrt{ax}+b)\,.\,(3\sqrt{ax}+5b)$,

$$3ax+2b\sqrt{ax}-5b^2=3ax+b\sqrt{ax}-2b^2;$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ b\sqrt{ax}=3b^2;$$

$$(18.\ \text{Cor. } 2.)\ \sqrt{ax}=3b;$$

$$(19)\ \text{squaring both sides, } ax=9b^2,$$

$$\text{and } (18.\ \text{Cor. } 2.)\ x=\frac{9b^2}{a}.$$

40. Given $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{\sqrt{3x}-1}{2}$, to find the value of x.

$$\text{Since } 3x-1=(\sqrt{3x}+1)\times(\sqrt{3x}-1);$$

$$\therefore\ \frac{3x-1}{\sqrt{3x}+1}=\sqrt{3x}-1;$$

$$\therefore\ \sqrt{3x}-1=1+\frac{\sqrt{3x}-1}{2},$$

and (17. Cor. 3.) taking $\frac{\sqrt{3x}-1}{2}$ from each side,

$$\frac{\sqrt{3x}-1}{2}=1;$$

$$\therefore\ (18.\ \text{Cor. } 1.)\ \sqrt{3x}-1=2;$$

$$\therefore\ (17)\ \text{by transposition, } \sqrt{3x}=2+1=3;$$

$$\therefore\ (19)\ \text{squaring both sides, } 3x=9,$$

$$(18.\ \text{Cor. } 2.)\ x=\frac{9}{3}=3.$$

41. Given $\frac{ax-b^2}{\sqrt{ax}+b}=c+\frac{\sqrt{ax}-b}{c}$, to find the value of x.

Since $ax - b^2 = (\sqrt{ax} + b) \cdot (\sqrt{ax} - b)$;

$$\therefore \frac{ax - b^2}{\sqrt{ax} + b} = \sqrt{ax} - b;$$

$$\therefore \sqrt{ax} - b = c + \frac{\sqrt{ax} - b}{c};$$

and taking $\frac{\sqrt{ax} - b}{c}$ from each side,

$$\frac{c-1}{c} \cdot (\sqrt{ax} - b) = c,$$

and (18. Cor. 1.) $\sqrt{ax} - b = \frac{c^2}{c-1}$;

by transposition, $\sqrt{ax} = b + \frac{c^2}{c-1}$;

(19) squaring both sides, $ax = \left(b + \frac{c^2}{c-1}\right)^2$.

$\therefore$ (18. Cor. 2.) $x = \frac{1}{a} \cdot \left(b + \frac{c^2}{c-1}\right)^2$.

42. Given $x = \sqrt{\{a^2 + x\sqrt{(b^2 + x^2)}\}} - a$, to find the value of x.

(17) by transposition, $x + a = \sqrt{\{a^2 + x\sqrt{(b^2 + x^2)}\}}$

$\therefore$ (19) squaring both sides, $x^2 + 2ax + a^2 = a^2 + x\sqrt{(b^2 + x^2)}$;

(17. Cor. 3.) $x^2 + 2ax = x\sqrt{(b^2 + x^2)}$;

(18. Cor. 2.) $x + 2a = \sqrt{(b^2 + x^2)}$;

and (19) squaring both sides, $x^2 + 4ax + 4a^2 = b^2 + x^2$

$\therefore$ (17. Cor. 3.) $4ax + 4a^2 = b^2$;

(17) by transposition, $4ax = b^2 - 4a^2$;

$\therefore$ (18. Cor. 2.) $x = \frac{b^2 - 4a^2}{4a}$.

43. Given $\sqrt{(2+x)} + \sqrt{x} = \frac{4}{\sqrt{(2+x)}}$ to find the value of x.

$$\text{(18. Cor. 1.) } 2 + x + \sqrt{(2x + x^2)} = 4;$$

$$\therefore \text{ (17) by transposition, } \sqrt{(2x + x^2)} = 4 - 2 - x = 2 - x,$$

$$\text{and (19) squaring both sides, } 2x + x^2 = 4 - 4x + x^2;$$

$$\therefore \text{ (17. Cor. 3.) } 6x = 4;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{4}{6} = \frac{2}{3}.$$

44. Given $\sqrt{(5+x)} + \sqrt{x} = \frac{15}{\sqrt{(5+x)}}$, to find the value of x.

$$\text{(18. Cor. 1.) } 5 + x + \sqrt{(5x + x^2)} = 15;$$

$$\therefore \text{ (17) by transposition, } \sqrt{(5x + x^2)} = 15 - 5 - x = 10 - x,$$

$$\text{and (19) squaring both sides, } 5x + x^2 = 100 - 20x + x^2;$$

$$\therefore \text{ (17. Cor. 3.) } 25x = 100;$$

$$\therefore \text{ (18. Cor. 2.) } x = \frac{100}{25} = 4.$$

45. Given $\sqrt{(x + \sqrt{x})} - \sqrt{(x - \sqrt{x})} = \frac{3}{2}\sqrt{\left(\frac{x}{x + \sqrt{x}}\right)}$, to find the value of x.

$$\text{(18. Cor. 1.) } x + \sqrt{x} - \sqrt{(x^2 - x)} = \frac{3\sqrt{x}}{2};$$

$$\therefore \text{ by transposition, } x - \frac{\sqrt{x}}{2} = \sqrt{(x^2 - x)},$$

$$\text{and (18. Cor. 2.) } \sqrt{x} - \tfrac{1}{2} = \sqrt{(x - 1)};$$

$$\text{(19) squaring both sides, } x - \sqrt{x} + \tfrac{1}{4} = x - 1;$$

$$\therefore \text{ (19. Cor. 3.) } \sqrt{x} = \frac{5}{4};$$

$$\text{and (19) squaring both sides, } x = \frac{25}{16}.$$

46. Given $\frac{1}{x} + \frac{1}{a} = \sqrt{\left\{\frac{1}{a^2} + \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)}\right\}}$, to find the value of x.

(19) squaring both sides,

$$\frac{1}{x^2} + \frac{2}{ax} + \frac{1}{a^2} = \frac{1}{a^2} + \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)};$$

$$\therefore \text{ (17. Cor. 3.) } \frac{1}{x^2} + \frac{2}{ax} = \sqrt{\left(\frac{4}{a^2x^2} + \frac{9}{x^4}\right)},$$

$$\text{and (18. Cor. 2.) } \frac{1}{x} + \frac{2}{a} = \sqrt{\left(\frac{4}{a^2} + \frac{9}{x^2}\right)};$$

$$\therefore \text{ (19) squaring both sides } \frac{1}{x^2} + \frac{4}{ax} + \frac{4}{a^2} = \frac{4}{a^2} + \frac{9}{x^2};$$

$$\text{(17. Cor. 3.) } \frac{4}{ax} = \frac{8}{x^2};$$

$$\text{(18. Cor. 2.) } \frac{1}{a} = \frac{2}{x};$$

$$\therefore \text{ (18. Cor. 1.) } x = 2a.$$

SECTION II.

On the Solution of Simple Equations which involve more than one unknown Quantity.

(23.) If the equation involve several unknown quantities, and definite values of these are required, there must necessarily be as many independent equations as there are unknown quantities. In which case, the values will be found by exterminating all the unknown quantities except one; and this may be done by either of the three following methods:

1. By equalizing the coefficients of the same unknown quantity in the several equations.
2. By substitution.
3. By equating different values of the same unknown quantity.

1. Of exterminating an unknown quantity by the first method in equations where two unknown quantities are concerned*.

If the coefficient of either unknown quantity in one equation be contained a certain number of times exactly in the coefficient of the same unknown quantity in the other, multiply the former equation by that number, then add it to, or subtract it from, the other equation, according as the signs are different or the same, and an equation arises, in which only one unknown quantity is found.

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* The first of these methods is also known by the name of the *method by addition and subtraction*, because the unknown quantities are exterminated by addition and subtraction, after the equations have been prepared in such a manner that one unknown quantity may have the same coefficient in each.

Thus, if $\left.\begin{array}{r} 4x + y = 34 \\ \text{and } 4y + x = 16 \end{array}\right\}$ Here the coefficient of x in the second equation is contained 4 times exactly in the first; multiplying therefore the second equation by 4, and subtracting the first from it,

$$\begin{aligned} 4x + 16y &= 64, \\ \text{and } 4x + y \quad &= 34; \\ \therefore\ 15y \quad &= 30, \text{ and } y = 2. \end{aligned}$$

Having thus obtained a value of one of the unknown quantities, the other may be determined by substituting in either equation the value of the quantity found, and thus reducing the equation to one which contains only the other unknown quantity. Thus, from the second of the preceding equations, $x = 16 - 4y = 16 - 8 = 8$.

The values of x and y might be found in a similar manner, by multiplying the first equation by 4, and subtracting the second from it.

But if neither of the coefficients be a measure of the coefficient of the same unknown quantity in the other equation, multiply the first equation by the coefficient of one of the unknown quantities in the second equation, and the second equation by the coefficient of the same unknown quantity in the first. If the signs of the unknown quantity be alike in both, subtract one equation from the other; if unlike, add them together, and an equation arises in which only one unknown quantity is found.

Thus, if $\left.\begin{array}{r} 2x + 3y = 23 \\ \text{and } 5x - 2y = 10 \end{array}\right\}$ In this case neither of the coefficients is a measure of the coefficient of the same unknown quantity in the other equation; and therefore, multiplying the first equation by 2, and the second by 3,

$$\begin{aligned} 4x + 6y &= 46, \\ \text{and } 15x - 6y &= 30; \\ \therefore \text{ by addition, } 19x &= 76, \text{ and } x = 4; \\ \text{whence, as before, } 3y &= 23 - 2x = 23 - 8 = 15, \\ \text{and } y &= 5. \end{aligned}$$

The values of x and y might also be obtained, by multiplying the first equation by 5, and the second by 2, and then subtracting the second from the first.

2. By substitution*.

Find the value of one of the unknown quantities, in terms of the other and known quantities, in the more simple of the two equations; and substitute this value instead of the quantity itself in the other equation; thus an equation is obtained in which there is only one unknown quantity.

Thus in the first of the preceding examples; from the second equation, $x = 16 - 4y$; substituting therefore this value of x in the first equation,

$$4 \,.\, (16 - 4y) + y = 34,$$
$$\text{or } 64 - 16y + y = 34;$$
$$\therefore \text{ by transposition, } (64 - 34 =)\ 30 = 15y,$$
$$\text{and therefore } 2 = y;$$
$$\text{whence, as before, } x = 8.$$

Here a value of x might have been obtained from the second equation, and substituted for it in the first; whence an equation would have arisen, involving only y; the value of which being found, that of x also might be determined, as before, by substitution.

Or a value of y might be determined from either equation, and substituted in the other; from which would arise an equation involving only x, the value of which might be found; and therefore the value of y also might be obtained by substitution.

Again, in the second example; from the first equation is obtained

$$2x = 23 - 3y; \text{ and therefore } x = \frac{23 - 3y}{2};$$

* There is an inconvenience attending the two latter methods, which the former does not offer, viz. they originate new equations with denominators, which must be got rid of. The method of substitution may be employed with advantage whenever the coefficient of one of the unknown quantities, as in the former of the preceding examples, is equal to unity. Here the inconvenience mentioned is not perceptible. But in general the first method is preferable.

substituting therefore this value in the second equation,

$$5 \cdot \frac{23-3y}{2} - 2y = 10,$$

$$\text{or } 115 - 15y - 4y = 20;$$

$$\therefore \text{ by transposition, } 115 - 20 = 15y + 4y,$$

$$\text{or } 95 = 19y;$$

$$\therefore\ 5 = y,$$

$$\text{and } x = \frac{23-3y}{2} = \frac{23-15}{2} = \frac{8}{2} = 4.$$

Here also a value of x might be obtained from the second equation, and substituted in the first, which would give an equation involving only y; or a value of y might be obtained from either equation, which substituted in the other would give an equation involving only x; the value of which might therefore be found, and consequently that of y might also be determined.

3. By equating different values of the same unknown quantity.

From each equation find the value of the same unknown quantity in terms of the other and known quantities; then, by equating the values so found, an equation arises containing only one unknown quantity.

Thus in the first of the preceding examples; from the first equation, $y = 34 - 4x$,

and from the second equation,

$$4y = 16 - x; \text{ and therefore } y = \frac{16-x}{4};$$

$$\therefore\ \frac{16-x}{4} = 34 - 4x;$$

$$\text{consequently, } 16 - x = 136 - 16x;$$

$$\therefore \text{ by transposition, } 16x - x = 136 - 16,$$

$$\text{or } 15x = 120;$$

$$\therefore\ x = 8,$$

$$\text{and } y = 34 - 4x = 34 - 32 = 2, \text{ as before.}$$

In this case also, two values of x are deducible from the two equations, which would give an equation involving y only; and the value of y being determined, that of x might also be found.

Again, in the second of the preceding examples;

$$\text{from the first equation, } x = \frac{23-3y}{2},$$

$$\text{and from the second, } x = \frac{10+2y}{5};$$

$$\therefore \frac{10+2y}{5} = \frac{23-3y}{2},$$

$$\text{and } 20 + 4y = 115 - 15y;$$

$$\text{by transposition, } 4y + 15y = 115 - 20,$$

$$\text{or } 19y = 95;$$

$$\therefore y = 5, \text{ and } x = 4, \text{ as before.}$$

Here again two values of y might have been found, which would have given an equation involving only x; and from the solution of this new equation, a value of x, and therefore of y, might be found.

EXAMPLES.

1. Given $\left.\begin{array}{l} 5x + 4y = 58 \\ \text{and } 3x + 7y = 67 \end{array}\right\}$ to find the values of x and y.

Multiplying the second equation by 5, and the first by 3,

$$15x + 35y = 335,$$

$$\text{and } 15x + 12y = 174;$$

$$\therefore \text{ by subtraction, } \quad 23y = 161,$$

$$\text{and } \quad y = 7;$$

$$\text{whence } 5x = 58 - 4y = 58 - 28 = 30,$$

$$\text{and therefore } x = 6.$$

If the second equation had been multiplied by 4, and subtracted from the first when multiplied by 7, an equation

would have arisen, involving only x, the value of which might be determined, and thence, by substitution, the value of y.

Second Method.

From the second equation, $3x = 67 - 7y$;

$$\therefore\ x = \frac{67 - 7y}{3}.$$

Substituting this value of x in the first equation,

$$5 \cdot \frac{67 - 7y}{3} + 4y = 58,$$

$$\text{and } 335 - 35y + 12y = 174;$$

$$\therefore \text{ by transposition, } 335 - 174 = 35y - 12y,$$

$$\text{or } 161 = 23y;$$

$$\therefore\ 7 = y;$$

whence, as before, the value of x may be found. In the same manner, a value of x might be found from the first equation, which substituted in the second, would give an equation involving only y. Or a value of y might be obtained from either equation, and substituted for it in the other; whence an equation would arise involving only x, the value of which might be found, and therefore that of y also determined.

Third Method.

From the first equation, $5x = 58 - 4y$;

$$\therefore\ x = \frac{58 - 4y}{5}.$$

$$\text{From the second, } x = \frac{67 - 7y}{3}.$$

$$\therefore\ \frac{58 - 4y}{5} = \frac{67 - 7y}{3},$$

$$\text{and } 174 - 12y = 335 - 35y;$$

$$\text{by transposition, } 35y - 12y = 335 - 174,$$

$$\text{or } 23y = 161;$$

$$\therefore\ y = 7;$$

whence, as before, $x = 6$.

In this case, two values of y might be deduced from the two equations; and from equating these, there would arise an equation involving x only; whose value being found, that of y also might be determined by substitution.

2. Given $\left.\begin{aligned} ax + by &= m \\ cx + dy &= n \end{aligned}\right\}$ to find the values of x and y.

Multiplying the first equation by c, and the second by a,

$$acx + bcy = mc,$$

$$acx + ady = na;$$

$\therefore$ by subtraction, $(ad - bc) \,.\, y = na - mc$,

$$\text{and } y = \frac{na - mc}{ad - bc};$$

$$\text{whence } x = \frac{m}{a} - \frac{by}{a} = \frac{m}{a} - \frac{nab - mbc}{a^2d - abc},$$

$$= \frac{mad - mbc}{a^2d - abc} - \frac{nab - mbc}{a^2d - abc},$$

$$= \frac{mad - nab}{a^2d - abc} = \frac{md - nb}{ad - bc}.$$

Or the value of x might be determined from the second equation, $x = \frac{n}{c} - \frac{dy}{c}$.

If the first equation had been multiplied by d, and subtracted from the second multiplied by b, an equation would have arisen involving only x, the value of which might be determined; and this being substituted in either of the equations, the value of y might also be found.

Second Method.

From the first equation, $ax = m - by$;

$$\text{and } \therefore\ x = \frac{m - by}{a}.$$

Substituting this value of x in the second equation,

$$c \cdot \frac{m - by}{a} + dy = n;$$

$$\therefore \; mc - bcy + ady = an,$$

$$\text{and } (ad - bc) \cdot y = an - mc;$$

$$\therefore \; y = \frac{an - mc}{ad - bc};$$

whence, the value of x may be determined, as before.

In the same manner, a value of x might be found from the second equation, which substituted in the first would give an equation involving only y, the value of which being found, that of x might also be determined. Or, a value of y might be obtained from either equation, which substituted in the other would give an equation involving only x, the value of which, and consequently that of y, might be found.

Third Method.

From the first equation, $x = \dfrac{m - by}{a}$,

and from the second, $x = \dfrac{n - dy}{c}$;

$$\therefore \; \frac{m - by}{a} = \frac{n - dy}{c};$$

$$\text{and } mc - bcy = na - ady;$$

$$\therefore \text{ by transposition, } ady - bcy = na - mc;$$

$$\therefore \; y = \frac{na - mc}{ad - bc};$$

whence, as before, $x = \dfrac{md - nb}{ad - bc}$.

In this case, two values of y might be deduced from the two equations; and from equating these, there would arise another equation involving only x, the value of which being determined, that of y also might be found by substitution.

3. Given $\left.\begin{array}{l} 11x + 3y = 100 \\ \text{and } \; 4x - 7y = 4 \end{array}\right\}$ to find the values of x and y.

Multiplying the first equation by 7, and the second by 3,

$$77x + 21y = 700,$$

$$\text{and } 12x - 21y = 12;$$

$$\therefore \text{ by addition, } \quad 89x = 712,$$

$$\text{and } x = 8;$$

$$\text{whence } 3y = 100 - 11x = 100 - 88 = 12;$$

$$\therefore \ y = 4.$$

4. Given $\left.\begin{array}{l}\dfrac{x}{2} + \dfrac{y}{3} = 7 \\ \text{and } \dfrac{x}{3} + \dfrac{y}{2} = 8\end{array}\right\}$ to find the values of x and y.

(18. Cor. 1.) clearing the equations of fractions, by multiplying each by 6,

$$3x + 2y = 42,$$

$$\text{and } 2x + 3y = 48;$$

and as the coefficients in this case are not aliquot parts, multiplying the first by 3, and the second by 2;

$$\therefore \ 9x + 6y = 126,$$

$$\text{and } 4x + 6y = 96;$$

$$\therefore \text{ by subtraction, } \quad 5x = 30,$$

$$\text{and } x = 6;$$

$$\text{whence, } 2y = 42 - 3x = 42 - 18 = 24,$$

$$\text{and } y = 12.$$

5. Given $\left.\begin{array}{l}\dfrac{x}{7} + 7y = 99 \\ \text{and } \dfrac{y}{7} + 7x = 51\end{array}\right\}$ to find the values of x and y.

(18. Cor. 1.) multiplying each equation by 7,

$$\therefore \ x + 49y = 693,$$

$$\text{and } 49x + y = 357;$$

$\therefore$ by addition, $50x + 50y = 1050$,

and $\therefore$ $x + y = 21$;

but since $x + 49y = 693$,

subtracting the upper equation from the lower,

$$48y = 672;$$

$$\therefore y = 14,$$

whence $x = 21 - y = 21 - 14 = 7$.

6. Given $\left.\begin{array}{l} \dfrac{x+2}{3} + 8y = 31 \\ \text{and } \dfrac{y+5}{4} + 10x = 192 \end{array}\right\}$ to find the values of x and y.

Clearing the first equation of fractions,

$$x + 2 + 24y = 93;$$

$\therefore$ by transposition, $x + 24y = 91$.

Clearing the second equation of fractions,

$$y + 5 + 40x = 768;$$

$\therefore$ by transposition, $40x + y = 763$.

Multiplying the first equation by 40, and subtracting the second from it,

$$\begin{aligned} 40x + 960y &= 3640; \\ 40x + y &= 763; \\ \therefore 959y &= 2877, \end{aligned}$$

and $y = 3$;

$\therefore$ $x = 91 - 24y = 91 - 72 = 19$.

7. Given $\left.\begin{array}{l} \dfrac{2x-y}{2} + 14 = 18 \\ \text{and } \dfrac{2y+x}{3} + 16 = 19 \end{array}\right\}$ to find the values of x and y.

By transposition, $\dfrac{2x-y}{2} = 4$, from the first equation,

and $\therefore$ $2x - y = 8$.

Also $\frac{2y + x}{3} = 3$, from the second equation,

$$\text{and } \therefore\ 2y + x = 9;$$

which, multiplied by 2, gives $2x + 4y = 18$;

$$\text{but } 2x - y = 8;$$

$$\therefore \text{ by subtraction, } \quad 5y = 10,$$

$$\text{and } y = 2,$$

$$\text{whence } x = 9 - 2y = 9 - 4 = 5.$$

8. Given $\left.\begin{aligned} \frac{2x + 3y}{6} + \frac{x}{3} &= 8 \\ \text{and } \frac{7y - 3x}{2} - y &= 11 \end{aligned}\right\}$. to find the values of x and y.

Clearing the first equation of fractions, $2x + 3y + 2x = 48$,

$$\text{or } 4x + 3y = 48;$$

and clearing the second of fractions, $7y - 3x - 2y = 22$,

$$\text{or } 5y - 3x = 22.$$

Multiplying this by 4, and the preceding one by 3,

$$9y + 12x = 144,$$

$$\text{and } 20y - 12x = 88;$$

$$\therefore \text{ by addition, } \quad 29y = 232,$$

$$\text{and } y = 8,$$

$$\text{whence } 4x = 48 - 3y = 48 - 24 = 24,$$

$$\text{and } x = 6.$$

9. Given $\left.\begin{aligned} 3x + \frac{7y}{2} &= 22 \\ \text{and } 11y - \frac{2x}{5} &= 20 \end{aligned}\right\}$ to find the values of x and y.

Clearing the first equation of fractions, $6x + 7y = 44$;

but from the second, $55y - 2x = 100$.

Multiplying this last by 3, $165y - 6x = 300$,

but $7y + 6x = 44$;

$\therefore$ by addition, $172y = 344$,

and $y = 2$.

Now $6x = 44 - 7y = 44 - 14 = 30$;

$\therefore x = 5$.

10. Given $\left.\begin{array}{l} x + 1 : y :: 5 : 3 \\ \text{and } \dfrac{2x}{3} - \dfrac{5-y}{2} = \dfrac{41}{12} - \dfrac{2x-1}{4} \end{array}\right\}$ to find the values of x and y.

From the second equation, (18. Cor. 1.) multiplied by 12,

$$8x - 30 + 6y = 41 - 6x + 3;$$

$\therefore$ by transposition, $14x + 6y = 74$,

and $7x + 3y = 37$.

But from the first equation, $5y = 3x + 3$,

or $5y - 3x = 3$.

Multiplying this equation by 7, $35y - 21x = 21$,

and the former by 3, $9y + 21x = 111$;

$\therefore$ by addition, $44y = 132$;

and $y = 3$;

$$\therefore x + 1 = \frac{5y}{3} = 5, \text{ and } x = 4.$$

11. Given $\left.\begin{array}{l} \dfrac{x-2}{5} - \dfrac{10-x}{3} = \dfrac{y-10}{4} \\ \text{and } \dfrac{2y+4}{3} - \dfrac{2x+y}{8} = \dfrac{x+13}{4} \end{array}\right\}$ to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 60,

$$12x - 24 - 200 + 20x = 15y - 150;$$

and by transposition, $32x - 15y = 74$.

Also (18. Cor. 1.) multiplying the second equation by 24,

$$16y + 32 - 6x - 3y = 6x + 78;$$

$\therefore$ by transposition, $13y - 12x = 46$.

Now the coefficients of x have aliquot parts; multiplying therefore this by 8, and the preceding by 3,

$$104y - 96x = 368,$$

and $96x - 45y = 222$;

$\therefore$ by addition, $59y = 590$; and $y = 10$;

and $32x = 15y + 74 = 150 + 74 = 224$;

$$\therefore x = 7.$$

12. Given $\left.\begin{array}{l} 2y - \dfrac{x+3}{4} = 7 + \dfrac{3x-2y}{5} \\ \text{and } 4x - \dfrac{8-y}{3} = 24\frac{1}{2} - \dfrac{2x+1}{2} \end{array}\right\}$ to find the values of x and y.

(18. Cor. 1.) from the first equation,

$$40y - 5x - 15 = 140 + 12x - 8y;$$

$\therefore$ by transposition, $48y - 17x = 155$,

and from the second equation,

$$24x - 16 + 2y = 147 - 6x - 3;$$

$\therefore$ by transposition, $30x + 2y = 160$.

Multiplying this by 24, $48y + 720x = 3840$;

but $48y - 17x = 155$;

$\therefore$ by subtraction, $737x = 3685$,

and $x = 5$,

and $2y = 160 - 30x = 160 - 150 = 10$;

$$\therefore y = 5.$$

13. Given $\left.\begin{array}{l} \dfrac{2y}{18} - \dfrac{8x-2}{36} = 1 - \dfrac{4+y}{3} + \dfrac{x-y}{6} \\ \text{and } x : 3y :: 4 : 7 \end{array}\right\}$ to find the values of x and y.

Reducing the first equation to lower terms,

$$\frac{y}{9} - \frac{4x-1}{18} = 1 - \frac{4+y}{3} + \frac{x-y}{6};$$

and therefore (18. Cor. 1.) multiplying by 18,

$$2y - 4x + 1 = 18 - 24 - 6y + 3x - 3y;$$

$\therefore$ by transposition, $7 = 7x - 11y$.

But from the second equation, $7x = 12y$.

Substituting therefore this value in the preceding equation,

$$7 = 12y - 11y = y,$$

and therefore $x = \frac{12y}{7} = 12$.

14. Given $x - \frac{3y-2+x}{11} = 1 + \frac{15x + \frac{4y}{3}}{33}$ and $\frac{3x+2y}{6} - \frac{y-5}{4} = \frac{11x+152}{12} - \frac{3y+1}{2}$ to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 33,

$$33x - 9y + 6 - 3x = 33 + 15x + \frac{4y}{3};$$

by transposition, $15x - 9y = 27 + \frac{4y}{3}$;

$\therefore$ $45x - 27y = 81 + 4y$,

and $45x - 31y = 81$.

(18. Cor. 1.) multiplying the second equation by 12,

$$6x + 4y - 3y + 15 = 11x + 152 - 18y - 6;$$

$\therefore$ by transposition, $19y - 5x = 131$.

Multiplying this by 9, $171y - 45x = 1179$;

but $45x - 31y = 81$;

$\therefore$ by addition, $140y = 1260$;

$\therefore$ $y = 9$,

and $5x = 19y - 131 = 171 - 131 = 40$;

$\therefore$ $x = 8$.

15. Given $\left.\begin{aligned} \frac{80+3x}{15} &= 18\frac{1}{3} - \frac{4x+3y-8}{7} \\ \text{and } 10y + \frac{6x-35}{5} &= 55 + 10x \end{aligned}\right\}$ to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 105, the least common multiple of 3, 7, and 15,

$$560 + 21x = 1925 - 60x - 45y + 120;$$

$\therefore$ by transposition, $81x + 45y = 1485$;

and dividing by 9, $9x + 5y = 165$.

From the second equation, $50y + 6x - 35 = 275 + 50x$;

$\therefore$ by transposition, $50y - 44x = 310$;

and dividing by 2, $25y - 22x = 155$;

multiplying the equation above, by 5, $25y + 45x = 825$;

$\therefore$ by subtraction, $67x = 670$,

and $x = 10$.

Now $5y = 165 - 9x = 165 - 90 = 75$;

$\therefore$ $y = 15$.

16. Given $\left.\begin{aligned} y + \frac{5x+2y}{6} - \frac{3y-12+8x}{5} &= 4 - \frac{15+2x-4y}{3} \\ \text{and } \frac{7x+13-5y}{4} + x &= 2y - \frac{3x+2y-16}{3} \end{aligned}\right\}$

to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 30, the least common multiple of 3, 5, and 6,

$$30y + 25x + 10y - 18y + 72 - 48x = 120 - 150 - 20x + 40y;$$

whence, by transposition, $102 = 18y + 3x$;

and dividing by 3, $34 = 6y + x$.

Clearing the second equation of fractions,

$$21x + 39 - 15y + 12x = 24y - 12x - 8y + 64;$$

and by transposition, $45x - 31y = 25$.

Multiplying the former by 45, $45x + 270y = 1530$;

$\therefore$ by subtraction, $301y = 1505$,

and $y = 5$;

whence $x = 34 - 6y = 34 - 30 = 4$.

17. Given $\left.\begin{array}{l} 1 + \dfrac{25 + 5y}{6} - \dfrac{7x - 6}{3} = 10 - \dfrac{3x - 10 + 7y}{12} \\ \text{and } \dfrac{12 - x}{9} : 5x - \dfrac{14 + y}{3} :: 1 : 8 \end{array}\right\}$

to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 12, the least common multiple of 3, 6, and 12,

$$12 + 50 + 10y - 28x + 24 = 120 - 3x + 10 - 7y;$$

$\therefore$ by transposition, $17y - 25x = 44$,

and (21) from the second equation,

$$\frac{96 - 8x}{9} = 5x - \frac{14 + y}{3};$$

$$\therefore\ 96 - 8x = 45x - 42 - 3y;$$

and by transposition, $138 = 53x - 3y$.

Multiplying this equation by 17, and the one found above by 3,

$$51y - 75x = 132,$$

and $-51y + 901x = 2346$;

$\therefore$ by addition, $826x = 2478$,

and $x = 3$.

Now $3y = 53x - 138 = 159 - 138 = 21$;

$$\therefore\ y = 7.$$

18. Given $\left.\begin{array}{l} \dfrac{4x}{x^2} + \dfrac{5y}{y^2} = \dfrac{9}{y} - 1 \\ \text{and } \dfrac{5}{x} + \dfrac{4}{y} = \dfrac{7}{x} + \dfrac{3}{2} \end{array}\right\}$ to find the values of x and y.

Reducing the first equation to lower terms,

$$\frac{4}{x} + \frac{5}{y} = \frac{9}{y} - 1;$$

$$\therefore \text{ by transposition, } \frac{4}{x} - \frac{4}{y} = -1;$$

from the second equation, by transposition, $-\frac{2}{x} + \frac{4}{y} = \frac{3}{2}$;

$$\therefore \text{ by addition, } \frac{2}{x} = \frac{1}{2};$$

$$\therefore 4 = x,$$

$$\text{and } \frac{4}{y} = \frac{4}{x} + 1 = 2;$$

$$\therefore 2y = 4, \text{ and } y = 2.$$

19. Given $\left.\begin{aligned} \frac{a}{x} + \frac{b}{y} &= m \\ \text{and } \frac{c}{x} + \frac{d}{y} &= n \end{aligned}\right\}$ to find the values of x and y.

Multiplying the first equation by c, and the second by a,

$$\frac{ac}{x} + \frac{bc}{y} = mc,$$

$$\text{and } \frac{ac}{x} + \frac{ad}{y} = na,$$

$$\therefore \text{ by subtraction, } (bc - ad) \cdot \frac{1}{y} = mc - na;$$

$$\therefore y = \frac{bc - ad}{mc - na},$$

$$\text{and } \frac{a}{x} = m - \frac{b}{y} = m - \frac{mbc - nab}{bc - ad}$$

$$= \frac{mbc - mad - mbc + nab}{bc - ad} = \frac{nab - mad}{bc - ad};$$

$$\therefore \frac{1}{x} = \frac{nb - md}{bc - ad},$$

$$\text{and } x = \frac{bc - ad}{nb - md}.$$

20. Given $x - \dfrac{2y - x}{23 - x} = 20 - \dfrac{59 - 2x}{2}$ and $y + \dfrac{y - 3}{x - 18} = 30 - \dfrac{73 - 3y}{3}$ to find the values of x and y.

Multiplying the first equation by 2,

$$2x - \frac{4y - 2x}{23 - x} = 40 - 59 + 2x;$$

$$\therefore \text{ by transposition, } 19 = \frac{4y - 2x}{23 - x},$$

$$\text{and } 437 - 19x = 4y - 2x;$$

$$\text{and by transposition, } 437 = 17x + 4y.$$

Also from the 2[d] equation, $3y + \dfrac{3y - 9}{x - 18} = 90 - 73 + 3y$,

$$\text{and (17. Cor. 3.) } \frac{3y - 9}{x - 18} = 17;$$

$$\therefore 3y - 9 = 17x - 306;$$

$$\text{by transposition, } 297 = 17x - 3y;$$

$$\text{but } 437 = 17x + 4y;$$

$$\therefore \text{ by subtraction, } 140 = 7y,$$

$$\text{and } 20 = y,$$

$$\text{and } 17x = 297 + 3y = 297 + 60 = 357;$$

$$\therefore x = 21.$$

21. Given $8x - \dfrac{16 + 60x}{3y - 1} = \dfrac{16xy - 107}{5 + 2y}$ and $2 + 6y + 9x = \dfrac{27x^2 - 12y^2 + 38}{3x - 2y + 1}$ to find the values of x and y.

Multiplying the first equation by $5 + 2y$,

$$40x + 16xy - \frac{80 + 300x + 32y + 120xy}{3y - 1} = 16xy - 107;$$

$$\therefore \text{ by trans}^n,\ 40x + 107 = \frac{80 + 300x + 32y + 120xy}{3y - 1};$$

and multiplying by $3y - 1$,

$$120xy - 40x + 321y - 107 = 80 + 300x + 32y + 120xy;$$

$$\therefore \text{ (17. Cor. 3.) } 289y - 340x = 187.$$

And from the second equation,

$$27x^2 - 12y^2 + 15x + 2y + 2 = 27x^2 - 12y^2 + 38;$$

$$\therefore \text{ by transposition, } 15x + 2y = 36;$$

whence, the coefficients of x having aliquot parts, multiplying the first equation by 3, and the second by 68,

$$867y - 1020x = 561,$$

$$\text{and } 136y + 1020x = 2448;$$

$$\therefore \text{ by addition, } 1003y = 3009,$$

$$\text{and } y = 3,$$

$$\text{and } 15x = 36 - 2y = 36 - 6 = 30;$$

$$\therefore x = 2.$$

22. Given $\left.\begin{array}{l} \dfrac{3x + 2y}{5} - \dfrac{5x - \frac{3y}{4} + 1}{3} = x + \dfrac{y - 2x}{10} - \dfrac{4x - y}{7} \\ \text{and } y + 2x : y - 2x :: 12x + 6y - 3 : 6y - 12x - 1 \end{array}\right\}$
to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 420,

$$252x + 168y - 700x + 105y - 140 = 420x + 42y - 84x - 240x + 60y,$$

$$\text{and by transposition, } 171y - 544x = 140.$$

From the second equation, (*Hind's Alg.* 179, 6.)

$$2y : 4x :: 12y - 4 : 24x - 2,$$

$$\text{and (}Alg.\text{ 179, 8.) } y : 2x :: 6y - 2 : 12x - 1;$$

$$\therefore \text{ (21) } 12xy - y = 12xy - 4x;$$

$$\text{(17. Cor. 3.) } y = 4x.$$

Which value of y being substituted in the first equation,

$$684x - 544x = 140,$$
$$\text{or } 140x = 140;$$
$$\therefore\ x = 1,$$
$$\text{and } y = 4x = 4.$$

23. Given $\left.\begin{aligned} 3 - \frac{7 + \frac{2x}{y}}{5} &= 5 - \frac{5x+9}{3y} \\ \text{and } y - \frac{4+15y}{6x-2} &= \frac{2xy - \frac{107}{8}}{2x+5} \end{aligned}\right\}$

to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by $15y$,

$$\therefore\ 45y - 21y - 6x = 75y - 25x - 45;$$

and by transposition, $51y - 19x = 45$.

Multiplying the second equation by $2x + 5$,

$$2xy + 5y - \frac{8x + 20 + 30xy + 75y}{6x - 2} = 2xy - \frac{107}{8};$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ 5y + \frac{107}{8} = \frac{8x + 20 + 30xy + 75y}{6x - 2};$$

and multiplying by $6x - 2$,

$$30xy - 10y + \frac{321x - 107}{4} = 8x + 20 + 30xy + 75y;$$

$$\therefore\ (17.\ \text{Cor. } 3.)\ \frac{321x - 107}{4} = 8x + 85y + 20,$$

$$\text{and } 321x - 107 = 32x + 340y + 80;$$

and by transposition, $-187 = 340y - 289x$.

The coefficients of y in this case having aliquot parts; multiplying the first by 20, and the last by 3,

$$1020y - 380x = 900,$$
$$\text{and } 1020y - 867x = -561;$$
$$\therefore \text{ by subtraction, } 487x = 1461,$$
$$\text{and } x = 3;$$

consequently, $51y = 45 + 19x = 45 + 57 = 102$;

$$\therefore\ y = 2.$$

(24.) If there be three unknown quantities, their values may be found from three independent equations.

For from two of the equations, a third, which involves only two of the unknown quantities, may be deduced by the preceding rules; and from the remaining equation, and one of the others, another which contains the same two unknown quantities. Having therefore two equations, which involve only two unknown quantities, these may be determined; and, by substituting their values in any of the original equations, that of the third quantity will be obtained. In some particular equations, two unknown quantities may be exterminated at once.

EXAMPLES.

1. Given $\left.\begin{aligned} x + y + z &= 31 \\ x + y - z &= 25 \\ x - y - z &= 9 \end{aligned}\right\}$ to find the values of x, y and z.

Adding the first and third equations, $2x = 40$;

$$\therefore\ x = 20.$$

Subtracting the second from the first, $2z = 6$;

$$\therefore\ z = 3;$$

and subtracting the third from the second, $2y = 16$;

$$\therefore\ y = 8.$$

2. Given $\left.\begin{aligned} x + y + z &= 29 \\ x + 2y + 3z &= 62 \\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} &= 10 \end{aligned}\right\}$ to find the values of x, y and z.

Subtracting the first equation from the second,

$$y + 2z = 33.$$

(18. Cor. 1.) multiplying the third equation by 12, the least common multiple of 2, 3, and 4,

$$6x + 4y + 3z = 120;$$

multiplying the first equation by 6, $6x + 6y + 6z = 174$;

$\therefore$ by subtraction, $2y + 3z = 54$;

but $2y + 4z = 66$;

$\therefore$ by subtraction, $z = 12$;

and $y = 33 - 2z = 33 - 24 = 9$;

also $x = 29 - y - z = 29 - 9 - 12 = 8$.

In like manner, had the first equation been multiplied by 2, and subtracted from the second, an equation would have resulted, involving only x and z; and had it been multiplied by 4, and subtracted from the third when cleared of fractions, another equation would have been obtained, involving also x and z; whence, by the preceding rules, the values of x and z would be found, and consequently the value of y also, by substitution. Or if the first equation be multiplied by 3, and the second subtracted from it, an equation would arise involving only x and y; and if the first, when multiplied by 3, be subtracted from the third when cleared of fractions, another would arise involving only x and y; whence the values of x and y might be determined. And hence the third, that of z, might be found.

Second Method.

From the first equation, $x = 29 - y - z$;

$\therefore$ substituting this value of x in the second equation,

$$29 - y - z + 2y + 3z = 62;$$

$\therefore$ by transposition, $y = 33 - 2z$.

Also substituting, in the third equation, the value of x found from the first,

$$\frac{29 - y - z}{2} + \frac{y}{3} + \frac{z}{4} = 10;$$

$\therefore$ (18. Cor. 1.) $174 - 6y - 6z + 4y + 3z = 120$,

and by transposition, $54 = 3z + 2y$;

in which, substituting the value of y found above,

$$54 = 3z + 66 - 4z;$$

$$\therefore \text{ by transposition, } z = 12;$$
$$\text{whence } y = 33 - 2z = 33 - 24 = 9,$$
$$\text{and } x = 29 - y - z = 29 - 9 - 12 = 8.$$

It may be observed, that there will be the same variety of solution, as in the last case, according as x, y, or z, is exterminated.

Third Method.

$$\text{From the first equation, } x = 29 - y - z,$$
$$\text{and from the second, } x = 62 - 2y - 3z;$$
$$\therefore 29 - y - z = 62 - 2y - 3z,$$
$$\text{and by transposition, } y = 33 - 2z.$$

$$\text{Again, from the third equation, } x = 20 - \frac{2y}{3} - \frac{z}{2};$$
$$\therefore 29 - y - z = 20 - \frac{2y}{3} - \frac{z}{2};$$
$$\text{and by transposition, } 9 - \frac{z}{2} = \frac{y}{3};$$
$$\therefore 27 - \frac{3z}{2} = y;$$
$$\text{whence } 27 - \frac{3z}{2} = 33 - 2z;$$
$$\therefore \text{ by transposition, } \frac{z}{2} = 6,$$
$$\text{and } z = 12;$$
$$\text{whence } y = 9, \text{ and } x = 8, \text{ as before.}$$

The same observation applies to this solution, as did to the last.

3. Given $\left.\begin{aligned} x + 2y + 3z &= 14 \\ 2x - 3y + 4z &= 8 \\ 3x + 4y - 5z &= -4 \end{aligned}\right\}$ to find the values of x, y and z.

Multiplying the first equation by 2,

$$2x + 4y + 6z = 28,$$
$$\text{but } 2x - 3y + 4z = 8;$$
$$\therefore \text{ by subtraction, } 7y + 2z = 20.$$

Again, multiplying the first equation by 3,

$$3x + 6y + 9z = 42,$$

$$\text{but } 3x + 4y - 5z = -4;$$

$$\therefore \text{ by subtraction, } 2y + 14z = 46,$$

$$\text{and } y + 7z = 23;$$

$$\therefore 7y + 49z = 161,$$

$$\text{but } 7y + 2z = 20;$$

$$\therefore \text{ by subtraction, } 47z = 141,$$

$$\text{and } z = 3;$$

$$\text{whence } y = 23 - 7z = 23 - 21 = 2,$$

$$\text{and } x = 14 - 2y - 3z = 14 - 4 - 9 = 1.$$

4. Given $\left.\begin{array}{l} bz + cy = a \\ az + cx = b \\ ay + bx = c \end{array}\right\}$ to find the values of x, y and z.

Multiplying the first equation by a, the second by b, and the third by c,

$$abz + acy = a^2,$$

$$abz + bcx = b^2,$$

$$acy + bcx = c^2;$$

$$\therefore \text{ by addition, } 2abz + 2acy + 2bcx = a^2 + b^2 + c^2,$$

$$\text{but } 2abz + 2acy = 2a^2;$$

$$\therefore \text{ by subtraction, } 2bcx = b^2 + c^2 - a^2,$$

$$\text{and } x = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\text{In the same way, } y = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\text{and } z = \frac{a^2 + b^2 - c^2}{2ab}.$$

5. Given
$$\left.\begin{aligned}\frac{4x+3y+z}{10}-\frac{2y+2z-x+1}{15}&=5+\frac{x-z-5}{5}\\ \text{and } \frac{9x+5y-2z}{12}-\frac{2x+y-3z}{4}&=\frac{7y+z+3}{11}+\frac{1}{6}\\ \text{and } \frac{5y+3z}{4}-\frac{2x+3y-z}{12}+2z&=y-1+\frac{3x+2y+7}{6}\end{aligned}\right\}$$
to find the values of x, y and z.

Multiplying the first equation by 30, the least common multiple of 5, 10, and 15,

$$12x+9y+3z-4y-4z+2x-2=150+6x-6z-30;$$
$$\therefore \text{ by transposition, } 8x+5y+5z=122.$$

Again, multiplying the second equation by 132, the least common multiple of 4, 6, 11, 12,

$$99x+55y-22z-66x-33y+99z=84y+12z+36+22;$$
$$\therefore \text{ by transposition, } 33x-62y+65z=58.$$

Again, multiplying the third equation by 12, the least common multiple of 12, 6, 4,

$$15y+9z-2x-3y+z+24z=12y-12+6x+4y+14;$$
$$\therefore \text{ by transposition, } 8x+4y-34z=-2;$$
$$\text{but from the first equation, } 8x+5y+5z=122;$$
$$\therefore \text{ by subtraction, } y+39z=124.$$

Also the third equation being divided by 2,
$$4x+2y-17z=-1.$$

Multiplying this by 33, and the second by 4,
$$132x+66y-561z=-33,$$
$$\text{and } 132x-248y+260z=232;$$
$$\therefore \text{ by subtraction, } 314y-821z=-265;$$
$$\text{but } 314y+12246z=38936$$
(by multiplying the equation found above by 314);
$$\therefore \text{ by subtraction, } 13067z=39201,$$
$$\text{and therefore } z=3;$$

$$\text{whence } y = 124 - 39z = 124 - 117 = 7,$$
$$\text{and } 4x = 17z - 2y - 1 = 51 - 14 - 1 = 36;$$
$$\therefore\ x = 9.$$

It is evident that any of the quantities x, y, z may be first exterminated. And the operation will be similar by the other two methods.

(25.) If there be four unknown quantities, their values may be found from four independent equations. For from the four given equations, by the preceding rules, three may be deduced which involve only three unknown quantities, the values of which may be found by the last Article; and hence the fourth may be found, by substituting in any of the four given equations, the values of the three quantities determined.

But it frequently happens that in equations involving four or more unknown quantities, we do not find each equation containing all the four. And in this case the labour of solution may frequently be abridged.

EXAMPLES.

1. Let $\left.\begin{aligned} 2x - 3y + 2z &= 13 \\ 4v - 2x &= 30 \\ 4y + 2z &= 14 \\ 5y + 3v &= 32 \end{aligned}\right\}$ to find the values of v, x, y and z.

Subtracting the first equation from the third,

$$7y - 2x = 1;$$

multiplying the second by 3, and the fourth by 4,

$$12v - 6x = 90,$$
$$12v + 20y = 128;$$
$$\therefore \text{ by subtraction, } 20y + 6x = 38,$$
$$\text{but } 21y - 6x = 3;$$
$$\therefore \text{ by addition } 41y = 41,$$
$$\text{and } y = 1.$$

$$\text{Also } 2x = 7y - 1 = 7 - 1 = 6,$$
$$\therefore\ x = 3;$$

$$\text{and } 4v = 2x + 30 = 36,$$
$$\therefore\ v = 9;$$
$$\text{and } 2z = 14 - 4y = 14 - 4 = 10,$$
$$\therefore\ z = 5.$$

2. Given $\left.\begin{array}{l} ax + by = a^2 \\ bx - az = b^2 \\ cx + dv = c^2 \\ dy - cz = d^2 \end{array}\right\}$ to find the values of v, x, y, z.

Multiplying the first equation by b, and the second by a,

$$abx + b^2y = a^2b,$$
$$\text{and } abx - a^2z = ab^2;$$
$$\therefore \text{ by subtraction, } b^2y + a^2z = ab\,.\,(a - b);$$
$$\therefore\ b^2dy + a^2dz = abd\,.\,(a - b);$$
$$\text{but from the fourth, } b^2dy - b^2cz = b^2d^2;$$
$$\therefore \text{ by subtraction, } (a^2d + b^2c)\,.\,z = a^2bd - ab^2d - b^2d^2,$$
$$\text{and } z = \frac{bd\,.\,(a^2 - ab - bd)}{a^2d + b^2c}.$$

Now from the second equation,

$$bx = b^2 + az = b^2 + \frac{abd\,.\,(a^2 - ab - bd)}{a^2d + b^2c},$$
$$= \frac{b\,.\,(a^3d + b^3c - abd^2)}{a^2d + b^2c};$$
$$\therefore\ x = \frac{a^3d + b^3c - abd^2}{a^2d + b^2c}.$$

Also from the third equation,

$$dv = c^2 - cx = c\,.\,(c - x),$$
$$= c\,.\,\frac{ad\,.\,(ac + bd - a^2) - b^2c\,.\,(b - c)}{a^2d + b^2c};$$
$$\text{and } v = \frac{acd\,.\,(ac + bd - a^2) - b^2c^2\,.\,(b - c)}{d\,.\,(a^2d + b^2c)}.$$

And from the fourth equation,

$$dy = d^2 + cz = d^2 + \frac{bcd.(a^2 - ab - bd)}{a^2d + b^2c},$$

$$= d.\frac{a^2d^2 + a^2bc - ab^2c}{a^2d + b^2c};$$

$$\text{and } \therefore\ y = \frac{a^2d^2 + a^2b^2 - ab^2c}{a^2d + b^2c}.$$

(26.) If there be n unknown quantities and n independent equations, the values of those quantities may be found in a similar manner. For from the n given equations, $n - 1$ may be deduced, involving only $n - 1$ unknown quantities; and from these $n - 1$, $n - 2$ may be obtained, involving only $n - 2$ unknown quantities; and so on, till only one equation remains, involving one unknown quantity; which being found, the values of all the rest may be determined by substitution.

(27.) If there be more unknown quantities than independent equations, some of these quantities cannot be found except in terms of the others; and by assuming values of these others, we may obtain an infinite number of corresponding values of the former quantities, which will satisfy the conditions proposed. See Sect. XI.

But if there be fewer unknown quantities than independent equations, the values of the unknown quantities may be found from the different equations; and if these values be the same, some of the equations are unnecessary; if different, the equations are incongruous.

SECTION III.

On the Solution of Pure Quadratics, and others which may be solved without completing the Square.

(28.) WHEN the terms of an equation involve the square of the unknown quantity only, the value of the square will be found by the preceding articles; and extracting the root on each side of the equation, the unknown quantity itself will be determined.

In the same way any pure equation may be solved; for the power of the unknown quantity standing alone on one side of the equation, the known quantities being transposed to the other, the simple unknown quantity will be determined by extracting the root.

And by the same process, any equation containing the powers of a function of the unknown quantity, or containing the powers of two unknown quantities, may frequently be reduced to lower dimensions.

EXAMPLES.

1. Given $x^2 - 17 = 130 - 2x^2$, to find the values of x.

By transposition, $3x^2 = 147$;

$\therefore\ x^2 = 49$,

and $x = \pm 7$.*

* The square root of a quantity may be either $+$ or $-$, and consequently all quadratic equations admit of two solutions. Thus, $+7 \times +7$, and -7×-7, are both equal to 49; and both, when substituted for x in the original equation, answer the conditions required. Every pure quadratic $\therefore$ will have two, and only two, roots which are equal in magnitude, but different in Algebraical sign.

2. Given $x^2 + ab = 5x^2$, to find the values of x.

By transposition, $ab = 4x^2$;

$$\therefore\ \pm\sqrt{ab} = 2x,$$

$$\text{and } \pm\frac{\sqrt{ab}}{2} = x.$$

3. Given $\left.\begin{array}{r} xy = a \\ \text{and } \frac{x}{y} = b \end{array}\right\}$, to find the values of x and y.

From the second equation, $x = by$.

Substituting this value in the first equation,

$$by^2 = a;$$

$$\therefore\ y^2 = \frac{a}{b},$$

and extracting the square root, $y = \pm\sqrt{\frac{a}{b}}$;

$$\therefore\ x = by = \pm b\sqrt{\frac{a}{b}} = \pm\sqrt{ab}.$$

4. Given $\left.\begin{array}{l} x + y : x :: 5 : 3 \\ \text{and } xy = 6 \end{array}\right\}$, to find the values of x and y.

Since $x + y : x :: 5 : 3$;

$\therefore$ (*Hind's Alg.* 179, 4.) $y : x :: 2 : 3$;

$\therefore$ (21) $3y = 2x$, and $y = \frac{2x}{3}$.

Substituting this value in the second equation,

$$\frac{2x^2}{3} = 6,$$

$$\text{and } x^2 = 9;$$

therefore, extracting the square root, $x = \pm 3$,

$$\text{whence } y = \frac{2x}{3} = \pm 2.$$

5. Given $\left.\begin{array}{l} x + y : x - y :: 3 : 1 \\ \text{and } x^3 - y^3 = 56 \end{array}\right\}$, to find the values of x and y.

From the first equation, (*Alg.* 179, 6.)

$$2x : 2y :: 4 : 2;$$

$$\therefore \ (\textit{Alg. } 179, 7.) \ x : y :: 2 : 1,$$

$$\text{and } x = 2y.$$

Substituting this value of x in the second equation,

$$\therefore \ 8y^3 - y^3 = 56,$$

$$\text{or } 7y^3 = 56;$$

$$\therefore \ y^3 = 8,$$

$$\text{and } y = 2,$$

$$\text{whence } x = 2y = 4.$$

6. Given $\left.\begin{array}{l} xy = a^2 \\ \text{and } x^2 + y^2 = s^2 \end{array}\right\}$, to find the values of x and y.

To the second equation, adding twice the first,

$$x^2 + 2xy + y^2 = s^2 + 2a^2;$$

$\therefore$ extracting the square root, $x + y = \pm\sqrt{(s^2 + 2a^2)}$;

and from the second, subtracting twice the first,

$$x^2 - 2xy + y^2 = s^2 - 2a^2;$$

$\therefore$ extracting the square root, $x - y = \pm\sqrt{(s^2 - 2a^2)}$;

$$\text{but } x + y = \pm\sqrt{(s^2 + 2a^2)};$$

$$\therefore \text{ by addition, } 2x = \pm\sqrt{(s^2 + 2a^2)} \pm \sqrt{(s^2 - 2a^2)},$$

$$\text{and } x = \pm \tfrac{1}{2}\{\sqrt{(s^2 + 2a^2)} + \sqrt{(s^2 - 2a^2)}\},$$

$$\text{by subtraction, } 2y = \pm\sqrt{(s^2 + 2a^2)} \mp \sqrt{(s^2 - 2a^2)},$$

$$\text{and } y = \pm \tfrac{1}{2}\{\sqrt{(s^2 + 2a^2)} - \sqrt{(s^2 - 2a^2)}\}.$$

7. Given $\left.\begin{array}{l} x - y : x :: 5 : 6 \\ \text{and } xy^2 = 384 \end{array}\right\}$, to find the values of x and y.

$$(\text{By } \textit{Alg. } 179, 4. \text{ and } 1.) \ x : y :: 6 : 1;$$

$$\therefore \ (21) \ x = 6y.$$

Substituting this value of x in the second equation,

$$6y^2 = 384;$$
$$\therefore\ y^2 = 64,$$
$$\text{whence } y = 4,$$
$$\text{and } x = 6y = 24.$$

8. Given $\left.\begin{array}{l} x + y : x :: 7 : 5 \\ \text{and } xy + y^2 = 126 \end{array}\right\}$, to find the values of x and y.

$$(Alg.\ 179,\ 4.)\ y : x :: 2 : 5;$$
$$\therefore\ (21)\ 2x = 5y,$$
$$\text{and } x = \frac{5y}{2}.$$

Substituting this value for x in the second equation,

$$\frac{5y^2}{2} + y^2 = 126;$$
$$\therefore\ 5y^2 + 2y^2 = 252,$$
$$\text{or } 7y^2 = 252;$$
$$\therefore\ y^2 = 36,$$
$$\text{and } y = \pm 6;$$
$$\therefore\ x = \frac{5y}{2} = \pm 15.$$

9. Given $\left.\begin{array}{l} (x + y)^2 : (x - y)^2 :: 64 : 1 \\ \text{and } xy = 63 \end{array}\right\}$, to find the values of x and y.

$$(Alg.\ 179,\ 9.)\ x + y : x - y :: 8 : 1;$$
$$\therefore\ (Alg.\ 179,\ 6.)\ 2x : 2y :: 9 : 7,$$
$$\text{and } (Alg.\ 179,\ 7.)\ x : y :: 9 : 7;$$
$$\therefore\ (21)\ 7x = 9y,$$
$$\text{and } x = \frac{9y}{7}.$$

Substituting this for x in the second equation,

$$\frac{9y^2}{7} = 63;$$

$$\therefore\ y^2 = 49,$$
$$\text{and } y = \pm 7;$$
$$\therefore\ x = \frac{9y}{7} = \pm 9.$$

10. Given $\left.\begin{array}{r} x^2 + xy = 12 \\ \text{and } y^2 + xy = 24 \end{array}\right\}$, to find the values of x and y.

Adding the two equations together,

$$x^2 + 2xy + y^2 = 36;$$
$$\therefore \text{ extracting the square root, } x + y = \pm 6.$$
$$\text{Now } x^2 + xy = x\,.\,(x+y) = \pm 6x;$$
$$\therefore\ \pm 6x = 12,$$
$$\text{and } x = \pm 2;$$
$$\text{and therefore } y = \pm 6 \mp 2 = \pm 4.$$

11. Given $\left.\begin{array}{r} x + y = s \\ \text{and } x^2 - y^2 = d^2 \end{array}\right\}$, to find the values of x and y.

$$\text{Since } x^2 - y^2 = (x+y)\,.\,(x-y) = s\,.\,(x-y);$$
$$\therefore\ s\,.\,(x-y) = d^2,$$
$$\text{and } x - y = \frac{d^2}{s};$$
$$\text{but } x + y = s;$$
$$\therefore \text{ by addition, } 2x = s + \frac{d^2}{s} = \frac{s^2 + d^2}{s},$$
$$\text{and } x = \frac{s^2 + d^2}{2s},$$
$$\text{and by subtraction, } 2y = s - \frac{d^2}{s} = \frac{s^2 - d^2}{s};$$
$$\therefore\ y = \frac{s^2 - d^2}{2s}.$$

12. Given $\left.\begin{array}{r} x + y = s \\ \text{and } \quad xy = a^2 \end{array}\right\}$, to find the values of x and y.

Squaring the first equation, $x^2 + 2xy + y^2 = s^2$,

and from the second, $4xy = 4a^2$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = s^2 - 4a^2$,

and extracting the square root, $x - y = \pm\sqrt{(s^2 - 4a^2)}$,

but $x + y = s$;

$\therefore$ by addition, $2x = s \pm \sqrt{(s^2 - 4a^2)}$,

and $x = \frac{1}{2}\{s \pm \sqrt{(s^2 - 4a^2)}\}$;

by subtraction, $2y = s \mp \sqrt{(s^2 - 4a^2)}$,

$\therefore$ $y = \frac{1}{2}\{s \mp \sqrt{(s^2 - 4a^2)}\}$.

13. Given $\left.\begin{array}{l} x + y = s \\ \text{and } x^2 + y^2 = a^2 \end{array}\right\}$, to find the values of x and y.

Squaring the first equation, $x^2 + 2xy + y^2 = s^2$,

and doubling the second, $2x^2 + 2y^2 = 2a^2$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 2a^2 - s^2$,

and extracting the square root, $x - y = \pm\sqrt{(2a^2 - s^2)}$,

but $x + y = s$;

$\therefore$ by addition, $2x = s \pm \sqrt{(2a^2 - s^2)}$;

and $x = \frac{1}{2}\{s \pm \sqrt{(2a^2 - s^2)}\}$;

also by subtraction, $2y = s \mp \sqrt{(2a^2 - s^2)}$;

$\therefore$ $y = \frac{1}{2}\{s \mp \sqrt{(2a^2 - s^2)}\}$.

14. Given $\left.\begin{array}{l} \sqrt[3]{x} + \sqrt[3]{y} = 5 \\ \text{and } \sqrt[3]{x} - \sqrt[3]{y} = 1 \end{array}\right\}$, to find the values of x and y.

Adding the two equations, $2\sqrt[3]{x} = 6$;

$\therefore$ $\sqrt[3]{x} = 3$,

and $x = 27$,

and subtracting the equations, $2\sqrt[3]{y} = 4$;

$\therefore$ $\sqrt[3]{y} = 2$,

and $y = 8$.

15. Given $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}$, to find the values of x.

$$\text{(18. Cor. 1.)}\ x\sqrt{(a^2 + x^2)} + a^2 + x^2 = 2a^2;$$

$$\text{by transposition, } x\sqrt{(a^2 + x^2)} = a^2 - x^2,$$

$$\text{and squaring both sides, } a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4;$$

$$\therefore\ 3a^2x^2 = a^4,$$

$$\text{and } x^2 = \frac{a^2}{3};$$

$$\therefore\ x = \pm\frac{a}{\sqrt{3}}.$$

16. Given $\sqrt{\left(\dfrac{a^2}{x^2} + b^2\right)} - \sqrt{\left(\dfrac{a^2}{x^2} - b^2\right)} = b$, to find the values of x.

$$\text{By transposition, } \sqrt{\left(\frac{a^2}{x^2} + b^2\right)} = \sqrt{\left(\frac{a^2}{x^2} - b^2\right)} + b;$$

$$\text{squaring both sides, } \frac{a^2}{x^2} + b^2 = \frac{a^2}{x^2} - b^2 + 2b\sqrt{\left(\frac{a^2}{x^2} - b^2\right)} + b^2;$$

$$\text{(17. Cor. 3.)}\ b^2 = 2b\sqrt{\left(\frac{a^2}{x^2} - b^2\right)},$$

$$\text{or } \frac{b}{2} = \sqrt{\left(\frac{a^2}{x^2} - b^2\right)};$$

$$\text{squaring both sides, } \frac{b^2}{4} = \frac{a^2}{x^2} - b^2;$$

$$\therefore\ \frac{5b^2}{4} = \frac{a^2}{x^2};$$

$$\text{and extracting the square root, } \pm\frac{\sqrt{5}\,.\,b}{2} = \frac{a}{x};$$

$$\therefore\ x = \pm\frac{2a}{\sqrt{5}\,.\,b}.$$

17. Given $\dfrac{a}{x} + \dfrac{\sqrt{(a^2 - x^2)}}{x} = \dfrac{x}{b}$, to find the values of x.

The given equation becomes $\frac{a}{x} + \sqrt{\left(\frac{a^2}{x^2} - 1\right)} = \frac{x}{b}$;

by transposition, $\sqrt{\left(\frac{a^2}{x^2} - 1\right)} = \frac{x}{b} - \frac{a}{x}$;

$\therefore$ squaring both sides, $\frac{a^2}{x^2} - 1 = \frac{x^2}{b^2} - \frac{2a}{b} + \frac{a^2}{x^2}$,

and by transposition, $\frac{2a}{b} - 1 = \frac{x^2}{b^2}$;

$\therefore 2ab - b^2 = x^2$,

and extracting the square root, $\pm \sqrt{(2ab - b^2)} = x$.

18. Given $\left.\begin{aligned} x^2 + y^2 &= \frac{13}{x-y} \\ \text{and} \quad xy &= \frac{6}{x-y} \end{aligned}\right\}$ to find the values of x and y.

From the first equation subtracting twice the second;

$$(x^2 - 2xy + y^2 =) \overline{x - y}|^2 = \frac{1}{x-y};$$

$$\therefore \ (x - y)^3 = 1, \text{ and } x - y = 1;$$

$$\therefore \ x^2 + y^2 = 13,$$

$$\text{and} \quad 2xy = 12;$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 25,$$

$$\text{and } x + y = \pm 5;$$

$$\text{but } x - y = \quad 1;$$

$$\therefore \text{ by addition, } \quad 2x \quad = \quad 6, \text{ or } -4,$$

$$\text{and } x = 3, \text{ or } -2,$$

$$\text{and by subtraction, } 2y = 4, \text{ or } -6;$$

$$\therefore \ y = 2, \text{ or } -3.$$

19. Given $\left.\begin{aligned} x^2 - xy &= 48y \\ \text{and } xy - y^2 &= \ 3x \end{aligned}\right\}$, to find the values of x and y.

Dividing the first equation by x, $x - y = \frac{48y}{x}$;

and the second by y, $x - y = \frac{3x}{y}$;

$$\therefore \frac{48y}{x} = \frac{3x}{y};$$

$$\therefore 48y^2 = 3x^2,$$

$$\text{and } 16y^2 = x^2;$$

$$\text{consequently, } \pm 4y = x;$$

$$\text{and first, suppose } + 4y = x;$$

$$\therefore (x - y =)\ 3y = \left(\frac{48y}{x} = \frac{48y}{4y} =\right) 12; \therefore y = 4;$$

$$\therefore x = 4y = 16.$$

But if $x = -4y$,

$$(x - y =) - 5y = \left(\frac{48y}{x} = \frac{48y}{-4y} =\right) - 12;$$

$$\therefore y = \frac{12}{5},$$

$$\text{and } x = -4y = -\frac{48}{5}.$$

20. Given $\left.\begin{array}{l} \dfrac{xy}{x^{\frac{1}{2}}y^{-\frac{1}{2}}} = 48 \\ \text{and } \dfrac{xy}{\sqrt{x}} = 24 \end{array}\right\}$, to find the values of x and y.

Dividing the first equation by the second,

$$\left(\frac{\sqrt{x}}{x^{\frac{1}{2}}y^{-\frac{1}{2}}} =\right) y^{\frac{1}{2}} = 2;$$

$$\therefore y = 4;$$

whence, from the second equation, $\frac{4x}{\sqrt{x}} = 4\sqrt{x} = 24$;

$$\therefore \sqrt{x} = 6,$$

$$\text{and } x = 36.$$

21. Given $\dfrac{x^2 + 3x - 7}{x + 2 + \dfrac{18}{x}} = 1$, to find the values of x.

Clearing the equation of fractions,

$$x^2 + 3x - 7 = x + 2 + \frac{18}{x};$$

$$\therefore \text{ by transposition, } x^2 + 2x = 9 + \frac{18}{x},$$

$$\text{or } x \,.\, (x + 2) = 9 \,.\, \left(1 + \frac{2}{x}\right) = \frac{9}{x} \,.\, (x + 2);$$

$$\therefore x = \frac{9}{x},$$

$$\text{and } x^2 = 9;$$

$$\therefore x = \pm 3.$$

22. Given $\sqrt{\left(\dfrac{x+a}{x}\right)} + 2\sqrt{\left(\dfrac{a}{x+a}\right)} = b^2 \,.\, \sqrt{\left(\dfrac{x}{x+a}\right)}$, to find the values of x.

Multiplying the equation by $\sqrt{\left(\dfrac{x + a}{x}\right)}$,

$$\frac{x + a}{x} + 2\sqrt{\frac{a}{x}} = b^2,$$

$$\text{or } 1 + \frac{a}{x} + 2\sqrt{\frac{a}{x}} = b^2,$$

$$\text{and extracting the square root, } 1 + \sqrt{\frac{a}{x}} = \pm b;$$

$$\text{by transposition, } \sqrt{\frac{a}{x}} = \pm b - 1,$$

$$\text{and squaring both sides, } \frac{a}{x} = (b \mp 1)^2;$$

$$\therefore x = \frac{a}{(b \mp 1)^2}.$$

23. Given $\dfrac{\sqrt{(a + x)}}{a} + \dfrac{\sqrt{(a + x)}}{x} = \dfrac{\sqrt{x}}{c}$, to find the values of x.

The equation by reduction becomes $\frac{a+x}{ax}.\sqrt{(a+x)} = \frac{\sqrt{x}}{c}$,

$$\text{and} \therefore \overline{a+x}|^{\frac{3}{2}} = \frac{a}{c}.x^{\frac{3}{2}};$$

$$\text{extracting the } (\tfrac{3}{2})^{\text{th}} \text{ root, } a + x = \overline{\frac{a}{c}}|^{\frac{2}{3}}.x;$$

$$\therefore \text{by transposition, } a = \left(\overline{\frac{a}{c}}|^{\frac{2}{3}} - 1\right).x,$$

$$\text{and } \frac{a}{\overline{\frac{a}{c}}|^{\frac{2}{3}} - 1} = x.$$

24. Given $\frac{(a+x)^{\frac{1}{n}}}{a} + \frac{(a+x)^{\frac{1}{n}}}{x} = \frac{x^{\frac{1}{n}}}{c}$, to find the value of x.

$$\text{The equation is } (a+x)^{\frac{1}{n}}.\left(\frac{1}{a} + \frac{1}{x}\right) = \frac{x^{\frac{1}{n}}}{c},$$

$$\text{or } (a+x)^{\frac{1}{n}}.\frac{a+x}{ax} = \frac{x^{\frac{1}{n}}}{c},$$

$$\text{or } (a+x)^{\frac{1}{n}+1} = \frac{a}{c}.x^{\frac{1}{n}+1};$$

$$\therefore \left(\frac{a+x}{x}\right)^{\frac{n+1}{n}} = \frac{a}{c};$$

$$\text{and extracting the root, } \frac{a+x}{x} = \overline{\frac{a}{c}}|^{\frac{n}{n+1}};$$

$$\text{or } \frac{a}{x} + 1 = \overline{\frac{a}{c}}|^{\frac{n}{n+1}};$$

$$\text{by transposition, } \frac{a}{x} = \overline{\frac{a}{c}}|^{\frac{n}{n+1}} - 1;$$

$$\therefore\ x = \frac{a}{\left(\frac{a}{c}\right)^{\frac{n}{n+1}} - 1};$$

25. Given $\frac{m}{n} \cdot x^{\frac{m}{n}-1} = \frac{r}{s} \cdot x^{\frac{r}{s}-1}$, to find the value of x.

(18. Cor. 2.) $\frac{m}{n} \cdot x^{\frac{m}{n}} = \frac{r}{s} \cdot x^{\frac{r}{s}}$;

$$\therefore\ \frac{x^{\frac{m}{n}}}{x^{\frac{r}{s}}} = \frac{nr}{ms},$$

$$\text{or } x^{\frac{ms-nr}{ns}} = \frac{nr}{ms};$$

$\therefore$ extracting the root, $x = \left(\frac{nr}{ms}\right)^{\frac{ns}{ms-nr}}$.

26. Given $\left.\begin{aligned} x^{\frac{2}{3}} + y^{\frac{2}{3}} &= 13 \\ \text{and } x^{\frac{1}{3}} + y^{\frac{1}{3}} &= 5 \end{aligned}\right\}$, to find the values of x and y.

$$\begin{aligned}
\text{Squaring the second equation,}\quad & x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 25; \\
\text{and doubling the first,}\quad & 2x^{\frac{2}{3}} \qquad\quad + 2y^{\frac{2}{3}} = 26; \\
\therefore \text{ by subtraction,}\quad & x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 1; \\
\text{and extracting the root,}\quad & x^{\frac{1}{3}} - y^{\frac{1}{3}} = \pm 1; \\
\text{but}\quad & x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5; \\
\therefore \text{ by addition,}\quad & 2x^{\frac{1}{3}} = 6, \text{ or } 4, \\
\text{and}\quad & x^{\frac{1}{3}} = 3, \text{ or } 2, \\
\text{whence}\quad & x = 27, \text{ or } 8, \\
\text{but by subtraction,}\quad & 2y^{\frac{1}{3}} = 4, \text{ or } 6, \\
\text{and}\quad & y^{\frac{1}{3}} = 2, \text{ or } 3; \\
& \therefore\ y = 8, \text{ or } 27.
\end{aligned}$$

27. Given $\left.\begin{aligned} xy^2 + y &= 21 \\ \text{and } x^2y^4 + y^2 &= 333 \end{aligned}\right\}$, to find the values of x and y.

$$\text{Squaring the first equation, } x^2y^4 + 2xy^3 + y^2 = 441;$$
$$\text{and doubling the second, } 2x^2y^4 \qquad + 2y^2 = 666;$$
$$\therefore \text{ by subtraction, } x^2y^4 - 2xy^3 + y^2 = 225;$$
$$\text{extracting the root, } xy^2 - y = \pm 15;$$
$$\text{but } xy^2 + y = \quad 21;$$
$$\therefore \text{ by subtraction, } 2y = 6, \text{ or } 36,$$
$$\text{and } y = 3, \text{ or } 18;$$
$$\text{but by addition, } 2xy^2 = 36, \text{ or } 6;$$
$$\therefore xy^2 = 18, \text{ or } 3,$$
$$\text{and therefore } x = \frac{18}{9}, \text{ or } \frac{3}{324},$$
$$= 2, \text{ or } \frac{1}{108}.$$

28. Given $\left.\begin{array}{l} x^2y + xy^2 = 180 \\ \text{and } x^3 + y^3 = 189 \end{array}\right\}$, to find the values of x and y.

Adding 3 times the first equation to the second,

$$x^3 + 3x^2y + 3xy^2 + y^3 = 729;$$

whence, extracting the cube root, $x + y = 9$.

$$\text{Now } x^2y + xy^2 = (x + y) \cdot xy = 180;$$
$$\therefore \text{ by substitution, } 9xy = 180,$$
$$\text{and } xy = 20.$$
$$\text{But } x^2 + 2xy + y^2 = 81,$$
$$\text{and } \qquad 4xy \qquad = 80;$$
$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1;$$
$$\therefore \text{ extracting the square root, } x - y = \pm 1,$$
$$\text{and } x + y = 9;$$
$$\therefore \text{ by addition, } 2x = 10, \text{ or } 8;$$
$$\therefore x = 5, \text{ or } 4;$$
$$\text{and by subtraction, } 2y = 8, \text{ or } 10;$$
$$\therefore y = 4, \text{ or } 5.$$

29. Given $\left.\begin{array}{r} x + \sqrt{xy} + y = 19 \\ \text{and } x^2 + xy + y^2 = 133 \end{array}\right\}$, to find the values of x and y.

Dividing the second equation by the first,

$$x - \sqrt{xy} + y = 7;$$

$$\text{but } x + \sqrt{xy} + y = 19;$$

$$\therefore \text{ by addition, } 2x + 2y = 26;$$

$$\text{and } x + y = 13;$$

$$\text{and by subtraction, } 2\sqrt{xy} = 12;$$

$$\therefore \sqrt{xy} = 6,$$

$$\text{and } xy = 36.$$

Now from the second equation, $x^2 + xy + y^2 = 133$;

and from the last, $3xy = 108$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 25$;

and extracting the square root, $x - y = \pm 5$;

but $x + y = 13$;

$\therefore$ by addition, $2x = 18$, or 8,

and $x = 9$, or 4;

but by subtraction, $2y = 8$, or 18,

and $y = 4$, or 9.

30. Given $\dfrac{a - \sqrt{(a^2 - x^2)}}{a + \sqrt{(a^2 - x^2)}} = b^2$, to find the values of x.

Since the value of a fraction is not altered if the numerator and denominator be multiplied by the same quantity. Multiply therefore by $a - \sqrt{(a^2 - x^2)}$,

$$\text{and } \frac{\{a - \sqrt{(a^2 - x^2)}\}^2}{x^2} = b^2.$$

Extracting the square root, $\dfrac{a - \sqrt{(a^2 - x^2)}}{x} = \pm b$,

$$\text{and } a - \sqrt{(a^2 - x^2)} = \pm b \cdot x;$$

$$\therefore \text{ by transposition, } a \mp b \,.\, x = \sqrt{(a^2 - x^2)};$$

$$\text{and squaring both sides, } a^2 \mp 2ab \,.\, x + b^2x^2 = a^2 - x^2;$$

$$\therefore \text{ by transposition, } (b^2 + 1) \,.\, x^2 = \pm\, 2ab \,.\, x;$$

$$\text{and therefore } x = \pm\, \frac{2ab}{1 + b^2}.$$

31. Given $\dfrac{\sqrt{x} + \sqrt{(x-a)}}{\sqrt{x} - \sqrt{(x-a)}} = \dfrac{n^2 a}{x-a}$, to find the values of x.

Multiplying the numerator and the denominator of the first fraction by $\sqrt{x} + \sqrt{x-a}$,

$$\frac{\{\sqrt{x} + \sqrt{(x-a)}\}^2}{a} = \frac{n^2 a}{x-a};$$

$$\{\sqrt{x} + \sqrt{(x-a)}\}^2 = \frac{n^2 a^2}{x-a};$$

$$\text{extracting the square root, } \sqrt{x} + \sqrt{(x-a)} = \pm\, \frac{na}{\sqrt{(x-a)}},$$

$$\text{and } \therefore \sqrt{(x^2 - ax)} + x - a = \pm\, na;$$

$$\text{by transposition, } \sqrt{(x^2 - ax)} = (1 \pm n) \,.\, a - x,$$

and squaring both sides,

$$x^2 - ax = (1 \pm n)^2 \,.\, a^2 - 2 \,.\, (1 \pm n) \,.\, ax + x^2;$$

$$\therefore \text{ by transposition, } (1 \pm 2n) \,.\, ax = (1 \pm n)^2 \,.\, a^2;$$

$$\therefore x = \frac{(1 \pm n)^2 \,.\, a}{1 \pm 2n}.$$

32. Given $\dfrac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} - \sqrt{(a-x)}} = b$, to find the values of x.

Multiplying the numerator and denominator by

$$\sqrt{(a+x)} + \sqrt{(a-x)}, \qquad \frac{\{\sqrt{(a+x)} + \sqrt{(a-x)}\}^2}{2x} = b,$$

$$\text{or } \frac{2a + 2\sqrt{(a^2 - x^2)}}{2x} = b;$$

$$\therefore \frac{a}{x} + \sqrt{\left(\frac{a^2}{x^2} - 1\right)} = b;$$

by transposition, $\sqrt{\left(\frac{a^2}{x^2} - 1\right)} = b - \frac{a}{x}$;

squaring both sides, $\frac{a^2}{x^2} - 1 = b^2 - \frac{2ab}{x} + \frac{a^2}{x^2}$;

$\therefore$ by transposition, $\frac{2ab}{x} = b^2 + 1$,

and $x = \frac{2ab}{b^2+1}$.

33. Given $\frac{a + x + \sqrt{(2ax + x^2)}}{a + x} = b$, to find the values of x.

The equation by reduction becomes $1 + \frac{\sqrt{(2ax + x^2)}}{a + x} = b$,

or $1 + \sqrt{\left\{1 - \left(\frac{a}{a+x}\right)^2\right\}} = b$;

by transposition, $\sqrt{\left\{1 - \left(\frac{a}{a+x}\right)^2\right\}} = b - 1$,

squaring both sides, $1 - \left(\frac{a}{a+x}\right)^2 = b^2 - 2b + 1$;

$\therefore$ (17. Cor. 1. and 3.) $\left(\frac{a}{a+x}\right)^2 = 2b - b^2$;

extracting the square root, $\frac{a}{a+x} = \pm\sqrt{(2b - b^2)}$,

and $\therefore$ $\frac{a+x}{a} = \pm\frac{1}{\sqrt{(2b - b^2)}}$;

$\therefore a + x = \pm\frac{a}{\sqrt{(2b - b^2)}}$,

and $x = \pm\frac{a}{\sqrt{(2b - b^2)}} - a = \pm a\left(\frac{1 \mp \sqrt{(2b - b^2)}}{\sqrt{(2b - b^2)}}\right)$.

34. Given $\frac{\sqrt{(a + x)}}{\sqrt{x}} + \frac{\sqrt{(a - x)}}{\sqrt{x}} = \sqrt{\frac{x}{b}}$, to find the values of x.

The equation by reduction, is

$$\sqrt{\left(\frac{a}{x}+1\right)} + \sqrt{\left(\frac{a}{x}-1\right)} = \sqrt{\frac{x}{b}};$$

by transposition, $\sqrt{\left(\frac{a}{x}+1\right)} = \sqrt{\frac{x}{b}} - \sqrt{\left(\frac{a}{x}-1\right)}$,

and squaring both sides,

$$\frac{a}{x}+1 = \frac{x}{b} - 2\sqrt{\frac{x}{b}} \cdot \sqrt{\left(\frac{a}{x}-1\right)} + \frac{a}{x} - 1;$$

$$\text{by transposition, } 2\sqrt{\left\{\frac{x}{b}\cdot\left(\frac{a}{x}-1\right)\right\}} = \frac{x}{b} - 2,$$

$$\text{and } \sqrt{\left\{\frac{x}{b}\cdot\left(\frac{a}{x}-1\right)\right\}} = \frac{x}{2b} - 1;$$

$$\text{squaring both sides, } \frac{a}{b} - \frac{x}{b} = \frac{x^2}{4b^2} - \frac{x}{b} + 1;$$

$$\therefore \text{ by transposition, } \frac{a}{b} - 1 = \frac{x^2}{4b^2};$$

$$\text{and } 4ab - 4b^2 = x^2;$$

$$\therefore \text{ extracting the square root, } \pm 2\sqrt{(ab - b^2)} = x.$$

85. Given $\left.\begin{array}{l} 3x - \frac{3x}{y} = y^2 - y \\ \text{and } y^2 + x = 4 \end{array}\right\}$, to find the values of x and y.

Reducing the first equation, $\frac{3x \cdot (y-1)}{y} = y \cdot (y-1)$;

$$\text{and therefore (18. Cor. 2.) } \frac{3x}{y} = y,$$

$$\text{and } 3x = y^2.$$

Substituting this value in the second equation, $4x = 4$;

$$\therefore x = 1,$$

$$\text{and } y^2 = 3x = 3;$$

$$\therefore y = \pm\sqrt{3}.$$

36. Given $\left.\begin{array}{l} x^2 + y\sqrt{xy} = 9 \\ \text{and } y^2 + x\sqrt{xy} = 18 \end{array}\right\}$, to find the values of x and y.

The first equation is $x^{\frac{3}{2}} \times (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 9$,

and the second, $y^{\frac{3}{2}} \times (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 18$.

Dividing the second by the first, $\dfrac{y^{\frac{3}{2}}}{x^{\frac{3}{2}}} = 2$.

$$\therefore\ y^{\frac{1}{2}} = 2x^{\frac{1}{2}},$$
$$\text{and } y = 4x.$$

Substituting this value in the first equation,

$$x^2 + 4x\sqrt{4x^2} = 9,$$
$$\text{or } 9x^2 = 9;$$
$$\therefore\ x = \pm 1;$$
$$\text{consequently, } y = 4x = \pm 4.$$

37. Given $\left.\begin{array}{l} \dfrac{x^2 + 2y^2}{\sqrt{y}} = x + 2xy \\ \text{and } x^2 - 2y^2 = 256 - x\sqrt{y} \end{array}\right\}$, to find the values of x and y.

From the first equation, $x^2 + 2y^2 = x\sqrt{y} + 2xy^{\frac{3}{2}}$;

$\therefore$ by transposition, $x^2 - x\sqrt{y} = 2xy^{\frac{3}{2}} - 2y^2$,

$$\text{or } x \,.\, (x - \sqrt{y}) = 2y^{\frac{3}{2}} \,.\, (x - \sqrt{y});$$
$$\therefore\ x = 2y^{\frac{3}{2}},$$
$$\text{and } x\sqrt{y} = 2y^2;$$

$\therefore$ adding these equals to the second equation,

$$x^2 = 256,$$
$$\text{and therefore } x = \pm 16;$$
$$\therefore\ 2y^{\frac{3}{2}} = \pm 16,$$
$$\text{and } y^{\frac{3}{2}} = \pm 8;$$
$$\therefore\ y^{\frac{1}{2}} = \pm 2,$$
$$\text{and } y = 4.$$

38. Given $\left.\begin{array}{l} x^2y + xy^2 = 30 \\ \text{and } \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6} \end{array}\right\}$, to find the values of x and y.

From the second equation, $x + y = \frac{5xy}{6}$;

but from the first, $xy = \frac{30}{x + y}$.

Substituting this value in the other equation,

$$x + y = \frac{5}{6} \times \frac{30}{x + y} = \frac{25}{x + y};$$

$$\therefore\ (x + y)^2 = 25;$$

and extracting the root, $x + y = \pm 5$.

Let $x + y = +5$; then $xy = +6$;

whence $x^2 + 2xy + y^2 = 25$,

and $4xy = 24$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 1$,

and $x - y = \pm 1$;

but $x + y = 5$;

$\therefore$ by addition, $2x = 6$, or 4; and $x = 3$, or 2;

by subtraction, $2y = 4$, or 6; and $y = 2$, or 3.

But if $x + y = -5$, then $xy = -6$;

whence $x^2 + 2xy + y^2 = 25$,

and $4xy = -24$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 49$;

and extracting the root, $x - y = \pm 7$;

but $x + y = -5$;

$\therefore$ by addition, $2x = 2$, or -12; and $x = 1$, or -6;

by subtraction, $2y = -12$, or 2; and $y = -6$, or 1.

39. Given $\left.\begin{array}{l} x^2y + xy^2 = 6 \\ \text{and } x^3y^2 + x^2y^3 = 12 \end{array}\right\}$, to find the values of x and y.

Dividing the second equation by the first, $xy = 2$.

But $x^2y + xy^2 = (x + y) \cdot xy = 6$,

$\therefore\ 2 \cdot (x + y) = 6$,

$$\text{and } x + y = 3;$$
$$\text{whence } x^2 + 2xy + y^2 = 9;$$
$$\text{but } \quad 4xy \quad = 8;$$
$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1;$$
$$\text{and extracting the root, } x - y = \pm 1;$$
$$\text{also } x + y = 3;$$
$$\therefore \text{ by addition, } 2x = 4, \text{ or } 2;$$
$$\therefore x = 2, \text{ or } 1,$$
$$\text{and by subtraction, } 2y = 2, \text{ or } 4;$$
$$\therefore y = 1, \text{ or } 2.$$

40. Given $\left.\begin{array}{l} x^3 - y^3 : (x-y)^3 :: 61 : 1 \\ \text{and } xy = 320 \end{array}\right\}$, to find the values of x and y.

$$\text{Since } x^3 - y^3 : x^3 - 3x^2y + 3xy^2 - y^3 :: 61 : 1;$$
$$\therefore (\textit{Alg. } 179, 4.)\ 3x^2y - 3xy^2 : (x-y)^3 :: 60 : 1,$$
$$\text{or } 3xy \times (x-y) : (x-y)^3 :: 60 : 1;$$

$\therefore$ (*Alg.* 179, 7.) $960 : (x-y)^2 :: 60 : 1$, dividing the first and second terms by $x - y$;

and $16 : (x-y)^2 :: 1 : 1$, dividing the first and third terms by 60;

$$\therefore (x-y)^2 = 16,$$
$$\text{and } x - y = \pm 4.$$
$$\text{But since } x^2 - 2xy + y^2 = 16,$$
$$\text{and } \quad 4xy \quad = 1280;$$
$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 1296;$$
$$\text{and extracting the root, } x + y = \pm 36;$$
$$\text{but } x - y = \pm 4;$$
$$\therefore \text{ by addition, } 2x = \pm 40, \text{ or } \pm 32;$$
$$\text{and } x = \pm 20, \text{ or } \pm 16;$$

but by subtraction, $2y = \pm 32$, or ± 40,

and $y = \pm 16$, or ± 20.

41. Given $\left.\begin{array}{r}(x^2 + y^2) \times (x + y) = 2336 \\ \text{and } (x^2 - y^2) \times (x - y) = 576\end{array}\right\}$, to find the values of x and y.

From the first equation, $x^3 + x^2y + xy^2 + y^3 = 2336$;

and from the second, $x^3 - x^2y - xy^2 + y^3 = 576$;

$\therefore$ by subtraction, $2x^2y + 2xy^2 = 1760$;

adding this to the first equation,

$$x^3 + 3x^2y + 3xy^2 + y^3 = 4096;$$

$\therefore$ extracting the cube root, $x + y = 16$,

and $2xy \,.\, (x + y) = 1760$,

or $16 \times 2xy = 1760$,

$\therefore \; xy = 55$.

Now $x^2 + 2xy + y^2 = 256$,

and $4xy = 220$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 36$,

and therefore $x - y = \pm 6$;

but $x + y = 16$;

$\therefore$ by addition, $2x = 22$, or 10,

and $x = 11$, or 5;

but by subtraction, $2y = 10$, or 22;

$\therefore \; y = 5$, or 11.

42. Given $\left.\begin{array}{l}x^3 + y^3 = (x + y) \,.\, xy \\ \text{and } x^2y + xy^2 = 4xy\end{array}\right\}$, to find the values of x and y.

Dividing the second equation by xy, $x + y = 4$;

$$\therefore\ x^3 + 3x^2y + 3xy^2 + y^3 = 64.$$

But from the first equation, $x^3 - x^2y - xy^2 + y^3 = 0$;

$\therefore$ by subtraction, $4x^2y + 4xy^2 = 64$;

$\therefore\ (x + y) \cdot xy = 16$,

and $xy = 4$.

But $x^2 + 2xy + y^2 = 16$,

and $4xy = 16$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 0$;

and extracting the root, $x - y = 0$;

but $x + y = 4$;

$\therefore$ by addition, $2x = 4$;

and $x = 2$;

but by subtraction, $2y = 4$;

$\therefore\ y = 2$.

43. Given $\left.\begin{array}{r}(x^2 + y^2) \times (x + y) = \dfrac{15xy}{2} \\ \text{and } (x^4 - y^4) \times (x^2 + y^2) = \dfrac{75x^2y^2}{4}\end{array}\right\}$, to find the values of x and y.

Dividing the second equation by the first,

$$(x^2 + y^2) \cdot (x - y) = \frac{5xy}{2}.$$

Again, dividing the first equation by this last,

$$\frac{x+y}{x-y} = 3;$$

$\therefore\ x+y = 3x - 3y$;

$\therefore$ by transposition, $4y = 2x$,

and $2y = x$;

whence $(x^2 + y^2) \cdot (x - y) = 5y^2 \times y = \dfrac{5y \times 2y}{2}$,

or $y = 1$;

and therefore $x = 2y = 2$.

44. Given $\left.\begin{array}{l}(x^2 - y^2) \times (x - y) = 3xy \\ \text{and } (x^4 - y^4) \times (x^2 - y^2) = 45x^2y^2\end{array}\right\}$, to find the values of x and y.

Dividing the second equation by the first,

$$(x^2 + y^2) \times (x + y) = 15xy,$$

$$\text{or } x^3 + x^2y + xy^2 + y^3 = 15xy;$$

$$\text{but from the first, } x^3 - x^2y - xy^2 + y^3 = 3xy;$$

$$\therefore \text{ by addition, } 2x^3 + 2y^3 = 18xy;$$

$$\text{and } x^3 + y^3 = 9xy;$$

$$\text{but by subtraction, } 2x^2y + 2xy^2 = 12xy;$$

$$\therefore \text{ dividing by } 2xy,\ x + y = 6;$$

$$\text{whence } x^3 + 3x^2y + 3xy^2 + y^3 = 216;$$

$$\text{but } x^3 + y^3 = 9xy;$$

$$\therefore \text{ by subtraction, } 3x^2y + 3xy^2 = 216 - 9xy,$$

$$\text{or } 3 \,.\, (x + y) \,.\, xy = 18xy = 216 - 9xy;$$

$$\therefore 27xy = 216,$$

$$\text{and } xy = 8.$$

$$\text{Now } x^2 + 2xy + y^2 = 36,$$

$$\text{and } 4xy = 32;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 4;$$

$$\text{and extracting the root, } x - y = \pm 2;$$

$$\text{but } x + y = 6;$$

$$\therefore \text{ by addition, } 2x = 8, \text{ or } 4;$$

$$\therefore x = 4, \text{ or } 2;$$

$$\text{and by subtraction, } 2y = 4, \text{ or } 8;$$

$$\therefore y = 2, \text{ or } 4.$$

SECTION IV.

Solution of Adfected Quadratics, involving only one unknown Quantity.

(29.) Let the terms be arranged on one side of the equation, according to the dimensions of the unknown quantity, beginning with the highest; and (17) the known quantities be transposed to the other side; then, if the square of the unknown quantity has any coefficient, either positive or negative, let all the terms be divided by this coefficient (13). If the square of half the coefficient of the second term be now added (11) to both sides of the equation*, that side which involves the unknown quantity will become a complete square; and (19) extracting the square root on both sides of the equation, a simple equation will be obtained, from which the values of the unknown quantity may be determined.

It may be observed, that all equations may be solved as quadratics, by completing the squares, in which there are two terms involving the unknown quantity or any function of it, and the index of one is double that of the other. Thus, $x^6 + px^3 = q$, $x^{2n} - px^n = q$, $x^{\frac{n}{2}} + x^{\frac{n}{4}} = a$, $a^2x^2 + ax = b$, $x^{3n} + ax^{\frac{3n}{2}} = b$, $p^2x^{4n} - px^{2n} = d$, $(x^2 + px + q)^2 + (x^2 + px + q) = r$, $x^2 \,.\, (x^2 + ax)^2 + bx \,.\, (x^2 + ax) = d$, are of the same form

* This is called completing the square; and that a complete square is thus obtained may be easily proved.

Let $x^2 \pm 2ax$ be the proposed quantity on one side, when the terms are arranged according to the form prescribed above; and suppose $y^2 =$ the quantity requisite to complete the square. Now the square of $x \pm d = x^2 \pm 2dx + d^2$, where it is evident that four times the product of the extreme terms is equal to the square of the middle term; and therefore, in order that $x^2 \pm 2ax + y^2$ may be a square, $4x^2y^2$ must be equal to $4a^2x^2$; therefore $y^2 = a^2 =$ the square of half the coefficient of the middle term.

as quadratics, and the value of the unknown quantity may be determined in the same manner. Many equations, also, in which more than one unknown quantity are involved, may in a similar manner be reduced to lower dimensions by completing the square, as $x^2y^2 + pxy = q$, $(x^2 + y^2)^2 + p \cdot (x^2 + y^2) = r$. Instances of this kind occur in the following

EXAMPLES.

1. Given $x^2 + 8x = 33$, to find the values of x.

Completing the square, $x^2 + 8x + 16 = 49$;

and extracting the root, $x + 4 = \pm 7$;

whence, by transposition, $x = 3$, or -11 *.

2. Given $x^2 + 6x + 4 = 59$, to find the values of x.

By transposition, $x^2 + 6x = 55$;

and completing the square, $x^2 + 6x + 9 = 64$;

$\therefore$ extracting the root, $x + 3 = \pm 8$;

whence $x = 5$, or -11.

3. Given $x^2 - 8x + 10 = 19$, to find the values of x.

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* A Quadratic Equation cannot have more than two distinct values of the unknown quantity which will satisfy it.

For, if possible, let the equation $ax^2 + px + q = 0$ have three distinct values of x, viz. α, β, γ;

$$\text{then } a\alpha^2 + p\alpha + q = 0,$$
$$a\beta^2 + p\beta + q = 0,$$
$$a\gamma^2 + p\gamma + q = 0.$$

If the first be subtracted from the second,

$$a \cdot (\alpha^2 - \beta^2) + p \cdot (\alpha - \beta) = 0,$$
$$\therefore a \cdot (\alpha + \beta) + p = 0;$$
$$\text{similarly } a \cdot (\alpha^2 - \gamma^2) + p \cdot (\alpha - \gamma) = 0,$$
$$\text{and } a \cdot (\alpha + \gamma) + p = 0.$$

Subtracting this from the former, $a \cdot (\beta - \gamma) = 0$; but a cannot be $= 0$; for then the proposed equation would not be a quadratic, $\therefore \beta - \gamma = 0$, or $\beta = \gamma$. A Quadratic Equation $\therefore$ has not *three* distinct values of the unknown quantity, but it may have two.

By transposition, $x^2 - 8x = 9$;

and completing the square, $x^2 - 8x + 16 = 25$;

$\therefore$ extracting the root, $x - 4 = \pm 5$,

and $x = 9$, or -1.

4. Given $x^2 - 2px = q$, to find the values of x.

Completing the square, $x^2 - 2px + p^2 = p^2 + q$;

extracting the root, $x - p = \pm \sqrt{(p^2 + q)}$;

$\therefore x = p \pm \sqrt{(p^2 + q)}$.

5. Given $x^2 + px = q$, to find the values of x.

Completing the square, $x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} + q$;

extracting the root, $x + \frac{p}{2} = \pm \sqrt{\left(\frac{p^2}{4} + q\right)}$;

$$\therefore x = -\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} + q\right)} = -\frac{p \pm \sqrt{(p^2 + 4q)}}{2}.$$

6. Given $x^2 - x + 3 = 45$, to find the values of x.

By transposition, $x^2 - x = 42$;

and completing the square, $x^2 - x + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4}$;

$\therefore$ extracting the root, $x - \frac{1}{2} = \pm \frac{13}{2}$,

and $x = 7$, or -6.

7. Given $3x^2 + 2x - 9 = 76$, to find the values of x.

By transposition and division, $x^2 + \frac{2}{3}x = \frac{85}{3}$;

and completing the square, $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{85}{3} + \frac{1}{9} = \frac{256}{9}$;

$\therefore$ extracting the root, $x + \frac{1}{3} = \pm \frac{16}{3}$;

whence $x = 5$, or $-\frac{17}{3}$.

8. Given $5x^2 - 4x + 3 = 159$, to find the values of x.

By transposition and division, $x^2 - \frac{4x}{5} = \frac{156}{5}$;

$\therefore$ completing the square, $x^2 - \frac{4x}{5} + \frac{4}{25} = \frac{156}{5} + \frac{4}{25} = \frac{784}{25}$;

and extracting the root, $x - \frac{2}{5} = \pm\frac{28}{5}$;

consequently, $x = 6$, or $-\frac{26}{5}$.

9. Given $ax^2 - bx = c$, to find the values of x.

By dividing each term by a, $x^2 - \frac{b}{a}\cdot x = \frac{c}{a}$;

completing the square, $x^2 - \frac{b}{a}\cdot x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} + \frac{c}{a}$

$$= \frac{b^2 + 4ac}{4a^2};$$

extracting the root, $x - \frac{b}{2a} = \pm\frac{\sqrt{(b^2 + 4ac)}}{2a}$;

$$\therefore x = \frac{b \pm \sqrt{(b^2 + 4ac)}}{2a}.$$

10. Given $6x + \frac{35 - 3x}{x} = 44$, to find the values of x.

(18. Cor. 1.) $6x^2 + 35 - 3x = 44x$;

$\therefore$ by transposition, $6x^2 - 47x = -35$;

and (18. Cor. 1.) $x^2 - \frac{47}{6}\cdot x = -\frac{35}{6}$;

therefore, completing the square,

$$x^2 - \frac{47}{6}\cdot x + \left(\frac{47}{12}\right)^2 = \frac{2209}{144} - \frac{35}{6} = \frac{1369}{144};$$

$\therefore$ extracting the root, $x - \frac{47}{12} = \pm\frac{37}{12}$,

and $x = 7$, or $\frac{5}{6}$.

11. Given $4x - \frac{14 - x}{x + 1} = 14$, to find the values of x.

Clearing the equation of fractions,

$$4x^2 + 4x - 14 + x = 14x + 14;$$

and therefore, by transposition, $4x^2 - 9x = 28$,

and (18. Cor. 1.) $x^2 - \frac{9}{4}x = 7$;

$\therefore$ completing the square, $\overline{x^2 - \frac{9}{4}x + \frac{9}{8}}\Big|^2 = 7 + \frac{81}{64} = \frac{529}{64}$;

and extracting the root, $x - \frac{9}{8} = \pm \frac{23}{8}$;

whence $x = 4$, or $-\frac{7}{4}$.

12. Given $3x - \frac{1121 - 4x}{x} = 2$, to find the values of x.

(18. Cor. 1.) $3x^2 - 1121 + 4x = 2x$;

$\therefore$ by transposition, $3x^2 + 2x = 1121$;

and (18. Cor. 1.) $x^2 + \frac{2}{3}x = \frac{1121}{3}$;

$\therefore$ completing the square, $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{1121}{3} + \frac{1}{9} = \frac{3364}{9}$;

and extracting the root, $x + \frac{1}{3} = \pm \frac{58}{3}$;

$\therefore x = 19$, or $-\frac{59}{3}$.

13. Given $\frac{8 - x}{2} - \frac{2x - 11}{x - 3} = \frac{x - 2}{6}$, to find the values of x.

Multiplying by 6, $24 - 3x - \frac{12x - 66}{x - 3} = x - 2$;

by transposition, $26 - 4x = \frac{12x - 66}{x - 3}$,

$$\text{and (18. Cor. 2.) } 13 - 2x = \frac{6x - 33}{x - 3};$$

$$\therefore\ 19x - 39 - 2x^2 = 6x - 33;$$

$\therefore$ changing signs, and transposing,

$$2x^2 - 13x = -6,$$

$$\text{and } x^2 - \frac{13}{2}x = -3;$$

and completing the square,

$$x^2 - \frac{13}{2}x + \left(\frac{13}{4}\right)^2 = \frac{169}{16} - 3 = \frac{121}{16};$$

$$\therefore \text{ extracting the root, } x - \frac{13}{4} = \pm\frac{11}{4};$$

$$\therefore\ x = 6, \text{ or } \frac{1}{2}.$$

14. Given $5x - \frac{3x - 3}{x - 3} = 2x + \frac{3x - 6}{2}$, to find the values of x.

$$\text{(18. Cor. 1.) } 10x^2 - 36x + 6 = 4x^2 - 12x + 3x^2 - 15x + 18;$$

$$\therefore \text{ by transposition, } 3x^2 - 9x = 12,$$

$$\text{and } x^2 - 3x = 4;$$

$$\therefore \text{ completing the square, } x^2 - 3x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4};$$

$$\text{and extracting the root, } x - \frac{3}{2} = \pm\frac{5}{2},$$

$$\text{and } x = 4, \text{ or } -1.$$

15. Given $\frac{16}{x} - \frac{100 - 9x}{4x^2} = 3$, to find the values of x.

$$\text{(18. Cor. 1.) } 64x - 100 + 9x = 12x^2;$$

$$\text{whence, by transposition, } 12x^2 - 73x = -100,$$

$$\text{and } x^2 - \frac{73}{12}x = -\frac{100}{12};$$

and completing the square,

$$x^2 - \frac{73}{12}.x + \left(\frac{73}{24}\right)^2 = \frac{5329}{576} - \frac{100}{12} = \frac{529}{576};$$

$$\therefore \text{ extracting the root, } x - \frac{73}{24} = \pm\frac{23}{24},$$

$$\text{and } x = 4, \text{ or } \frac{25}{12}.$$

16. Given $3x - \dfrac{169 - 3x}{x} = 29$, to find the values of x.

$$\text{Here } 3x^2 - 169 + 3x = 29x;$$

$$\therefore \text{ by transposition, } 3x^2 - 26x = 169,$$

$$\text{and } x^2 - \frac{26}{3}x = \frac{169}{3};$$

therefore, completing the square,

$$x^2 - \frac{26}{3}x + \frac{169}{9} = \frac{169}{3} + \frac{169}{9} = \frac{4}{9}.169;$$

$$\text{and extracting the root, } x - \frac{13}{3} = \pm\frac{2}{3}.13;$$

$$\therefore x = 13, \text{ or } -\frac{13}{3}.$$

17. Given $16 - \dfrac{2x^2}{3} = \dfrac{4x}{5} + 7\dfrac{3}{5}$, to find the values of x.

$$\text{Multiplying every term by } \frac{3}{2},\ 24 - x^2 = \frac{6x}{5} + \frac{57}{5};$$

$\therefore$ (17. Cor. 1.) and by transposition,

$$x^2 + \frac{6x}{5} = 24 - \frac{57}{5} = \frac{63}{5};$$

and completing the square,

$$x^2 + \frac{6x}{5} + \frac{9}{25} = \frac{63}{5} + \frac{9}{25} = \frac{324}{25};$$

$\therefore$ extracting the root, $x + \frac{3}{5} = \pm \frac{18}{5}$,

and $x = 3$, or $-\frac{21}{5}$.

18. Given $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9}$, to find the values of x.

(18. Cor. 1) $10x - 14 + 2x = \frac{22x^2}{9}$;

$\therefore$ (17. Cor. 1.) $\frac{22x^2}{9} - 12x = -14$,

and $x^2 - \frac{54}{11}x = -\frac{63}{11}$;

$\therefore$ completing the square,

$$x^2 - \frac{54}{11}x + \left(\frac{27}{11}\right)^2 = \frac{729}{121} - \frac{63}{11} = \frac{36}{121};$$

and extracting the root, $x - \frac{27}{11} = \pm \frac{6}{11}$;

$\therefore$ $x = 3$, or $\frac{21}{11}$.

19. Given $\frac{3x - 4}{x - 4} + 1 = 10 - \frac{x - 2}{2}$, to find the values of x.

Clearing the equation of fractions,

$$6x - 8 + 2x - 8 = 20x - 80 - x^2 + 6x - 8;$$

$\therefore$ by transposition, $x^2 - 18x = -72$;

and completing the square, $x^2 - 18x + 81 = 81 - 72 = 9$;

whence, extracting the root, $x - 9 = \pm 3$;

and therefore $x = 12$, or 6.

20. Given $\frac{3x + 4}{5} - \frac{30 - 2x}{x - 6} = \frac{7x - 14}{10}$, to find the values of x.

Multiplying by 10, $6x + 8 - \frac{300 - 20x}{x - 6} = 7x - 14$;

$$\therefore \text{ by transposition, } 22 - x = \frac{300 - 20x}{x - 6};$$

$$\text{and } 28x - x^2 - 132 = 300 - 20x;$$

$$\therefore \text{ by transposition, and changing signs, } x^2 - 48x = -432;$$

$$\text{completing the square, } x^2 - 48x + 576 = 576 - 432 = 144;$$

$$\therefore \text{ extracting the root, } x - 24 = \pm 12;$$

$$\therefore x = 36, \text{ or } 12.$$

21. Given $3x - \frac{3x - 10}{9 - 2x} = 2 + \frac{6x^2 - 40}{2x - 1}$, to find the values of x.

Multiplying by $2x - 1$,

$$6x^2 - 3x - \frac{6x^2 - 23x + 10}{9 - 2x} = 4x - 2 + 6x^2 - 40,$$

$$\text{or } 7x + \frac{6x^2 - 23x + 10}{9 - 2x} = 42;$$

$$\therefore 63x - 14x^2 + 6x^2 - 23x + 10 = 378 - 84x;$$

$$\text{by transposition, } 124x - 8x^2 = 368,$$

$$\text{and } x^2 - \frac{31}{2}x = -46;$$

$\therefore$ completing the square,

$$x^2 - \frac{31}{2}x + \frac{961}{16} = \frac{961}{16} - 46 = \frac{225}{16};$$

$$\therefore \text{ extracting the root, } x - \frac{31}{4} = \pm \frac{15}{4};$$

$$\text{and therefore } x = \frac{23}{2}, \text{ or } 4.$$

22. Given $\frac{x}{5 + x} + \frac{7}{6 - 4x} = \frac{11x}{11x - 8}$, to find the values of x.

Multiplying by $(5 + x)$, $x + \frac{35 + 7x}{6 - 4x} = \frac{55x + 11x^2}{11x - 8}$;

and multiplying by $11x - 8$,

$$11x^2 - 8x + \frac{329x + 77x^2 - 280}{6 - 4x} = 55x + 11x^2,$$

$$\text{or } \frac{329x + 77x^2 - 280}{6 - 4x} = 63x;$$

$$\text{or, dividing by } 7, \frac{47x + 11x^2 - 40}{6 - 4x} = 9x;$$

$$\therefore\ 47x + 11x^2 - 40 = 54x - 36x^2;$$

by transposition, $47x^2 - 7x = 40$;

$$\therefore\ x^2 - \frac{7}{47}x = \frac{40}{47};$$

and completing the square,

$$x^2 - \frac{7}{47}x + \left(\frac{7}{94}\right)^2 = \frac{49}{8336} + \frac{40}{47} = \frac{7569}{8336};$$

$$\text{and extracting the root, } x - \frac{7}{94} = \pm\frac{87}{94};$$

$$\therefore\ x = 1, \text{ or } -\frac{40}{47}.$$

23. Given $\frac{90}{x} - \frac{27}{x+2} = \frac{90}{x+1}$, to find the values of x.

Dividing every term by 9, $\frac{10}{x} - \frac{3}{x+2} = \frac{10}{x+1}$;

$\therefore$ (18. Cor. 1.) $10x^2 + 30x + 20 - 3x^2 - 3x = 10x^2 + 20x$;

$\therefore$ by transposition and (17. Cor. 1.) $3x^2 - 7x = 20$,

$$\text{and } x^2 - \frac{7}{3}x = \frac{20}{3};$$

$$\text{completing the square, } x^2 - \frac{7}{3}x + \frac{49}{36} = \frac{20}{3} + \frac{49}{36} = \frac{289}{36};$$

$$\text{and extracting the root, } x - \frac{7}{6} = \pm\frac{17}{6};$$

$$\therefore\ x = 4, \text{ or } -\frac{5}{3}.$$

24. Given $\frac{1}{x^2 - 3x} + \frac{1}{x^2 + 4x} = \frac{9}{8x}$, to find the values of x.

Multiplying every term by x, $\frac{1}{x-3} + \frac{1}{x+4} = \frac{9}{8}$;

$\therefore$ (18. Cor. 1.) $8x + 32 + 8x - 24 = 9x^2 + 9x - 108$;

$\therefore$ by transposition and (17. Cor. 1.) $9x^2 - 7x = 116$,

$$\text{and } x^2 - \frac{7}{9}x = \frac{116}{9};$$

completing the square,

$$x^2 - \frac{7}{9}x + \frac{49}{324} = \frac{116}{9} + \frac{49}{324} = \frac{4225}{324};$$

$$\therefore \text{ extracting the root, } x - \frac{7}{18} = \pm\frac{65}{18};$$

$$\therefore\ x = 4, \text{ or } -\frac{29}{9}.$$

25. Given $\frac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3$, to find the values of x.

(18. Cor. 1.) $x^3 - 10x^2 + 1 = x^3 - 9x^2 + 27x - 27$;

$\therefore$ by transposition and (17. Cor. 1.) $x^2 + 27x = 28$;

and completing the square,

$$x^2 + 27x + \left(\frac{27}{2}\right)^2 = 28 + \frac{729}{4} = \frac{841}{4};$$

$$\therefore \text{ extracting the root, } x + \frac{27}{2} = \pm\frac{29}{2},$$

$$\text{and } x = 1, \text{ or } -28.$$

26. Given $\frac{x}{7-x} + \frac{7-x}{x} = 2\frac{9}{10}$, to find the values of x.

Clearing the equation of fractions,

$$20x^2 - 140x + 490 = 203x - 29x^2;$$

$$\therefore \text{ by transposition, } 49x^2 - 343x = -490,$$

$$\text{and } x^2 - 7x = -10;$$

$\therefore$ completing the square, $x^2 - 7x + \frac{49}{4} = \frac{49}{4} - 10 = \frac{9}{4}$;

and extracting the root, $x - \frac{7}{2} = \pm \frac{3}{2}$;

$\therefore \ x = 5$, or 2.

27. Given $\frac{7 - 12x}{x^{\frac{3}{2}}} = \frac{x}{\sqrt{x}} - \frac{8x + 110}{\sqrt{x^3}}$, to find the values of x.

Clearing the equation of fractions,

$$7 - 12x = x^2 - 8x - 110;$$

$$\therefore \ 117 = x^2 + 4x;$$

and completing the square, $121 = x^2 + 4x + 4$;

$\therefore$ extracting the root, $\pm 11 = x + 2$;

and therefore $x = 9$, or -13.

28. Given $\sqrt{(x + 5)} \times \sqrt{(x + 12)} = 12$, to find the values of x.

Squaring both sides, $(x + 5) \cdot (x + 12) = 144$;

or $x^2 + 17x + 60 = 144$;

$\therefore$ by transposition, $x^2 + 17x = 84$;

and completing the square,

$$x^2 + 17x + \left(\frac{17}{2}\right)^2 = \frac{289}{4} + 84 = \frac{625}{4};$$

extracting the root, $x + \frac{17}{2} = \pm \frac{25}{2}$,

and $x = 4$, or -21.

29. Given $\sqrt[3]{(x^3 - a^3)} = x - b$, to find the values of x.

Cubing each side, $x^3 - a^3 = x^3 - 3bx^2 + 3b^2x - b^3$;

$\therefore$ by transposition, $3bx^2 - 3b^2x = a^3 - b^3$,

and $x^2 - bx = \frac{a^3 - b^3}{3b}$;

∴ completing the square,

$$x^2 - bx + \frac{b^2}{4} = \frac{a^3 - b^3}{3b} + \frac{b^2}{4} = \frac{4a^3 - b^3}{12b};$$

$$\text{extracting the root, } x - \frac{b}{2} = \pm\sqrt{\left(\frac{4a^3 - b^3}{12b}\right)},$$

$$\text{and } x = \frac{b}{2} \pm \sqrt{\left(\frac{4a^3 - b^3}{12b}\right)}.$$

30. Given $x^{2n} - mx^n = p$, to find the values of x.

Completing the square,

$$x^{2n} - mx^n + \frac{m^2}{4} = \frac{m^2}{4} + p = \frac{m^2 + 4p}{4};$$

$$\text{extracting the root, } x^n - \frac{m}{2} = \pm\frac{\sqrt{(m^2 + 4p)}}{2};$$

$$\therefore\ x^n = \frac{m \pm \sqrt{(m^2 + 4p)}}{2},$$

$$\text{and } x = \left(\frac{m \pm \sqrt{(m^2 + 4p)}}{2}\right)^{\frac{1}{n}}.$$

31. Given $\dfrac{\sqrt{4x} + 2}{4 + \sqrt{x}} = \dfrac{4 - \sqrt{x}}{\sqrt{x}}$, to find the values of x.

Clearing the equation of fractions, $2x + 2\sqrt{x} = 16 - x$;

∴ by transposition, $3x + 2\sqrt{x} = 16$,

$$\text{and } x + \frac{2}{3}\sqrt{x} = \frac{16}{3};$$

$$\text{completing the square } x + \frac{2}{3}\sqrt{x} + \frac{1}{9} = \frac{16}{3} + \frac{1}{9} = \frac{49}{9};$$

$$\text{and extracting the root, } \sqrt{x} + \frac{1}{3} = \pm\frac{7}{3};$$

$$\therefore\ \sqrt{x} = 2, \text{ or } -\frac{8}{3},$$

$$\text{and } x = 4, \text{ or } \frac{64}{9}.$$

32. Given $\frac{\sqrt{a^2x} + b}{a + \sqrt{x}} = \frac{a - \sqrt{x}}{\sqrt{x}}$, to find the values of x.

Clearing the equation of fractions, $ax + b\sqrt{x} = a^2 - x$;

by transposition, $(a + 1) \cdot x + b\sqrt{x} = a^2$,

$$\text{and } x + \frac{b}{a+1} \cdot \sqrt{x} = \frac{a^2}{a+1};$$

completing the square,

$$x + \frac{b}{a+1} \cdot \sqrt{x} + \frac{b^2}{4 \cdot (a+1)^2} = \frac{a^2}{a+1} + \frac{b^2}{4 \cdot (a+1)^2} = \frac{4a^3 + 4a^2 + b^2}{4 \cdot (a+1)^2},$$

and extracting the root,

$$\sqrt{x} + \frac{b}{2 \cdot (a+1)} = \pm \frac{\sqrt{(4a^3 + 4a^2 + b^2)}}{2 \cdot (a+1)};$$

$$\therefore x = \left\{\frac{-b \pm \sqrt{(4a^3 + 4a^2 + b)}}{2 \cdot (a+1)}\right\}^2.$$

33. Given $\sqrt{x^3} - 2\sqrt{x} - x = 0$, to find the values of x.

Dividing by $\sqrt{x}$, we find $x - 2 - \sqrt{x} = 0$;

$\therefore$ by transposition, $x - \sqrt{x} = 2$;

and completing the square, $x - \sqrt{x} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$;

extracting the root, $\sqrt{x} - \frac{1}{2} = \pm \frac{3}{2}$;

$\therefore \sqrt{x} = 2$, or -1,

and $x = 4$, or 1.

34. Given $\sqrt{x^5} + \sqrt{x^3} = 6\sqrt{x}$, to find the values of x.

Dividing by $\sqrt{x}$, $x^2 + x = 6$;

$\therefore$ completing the square, $x^2 + x + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

and extracting the root, $x + \frac{1}{2} = \pm \frac{5}{2}$,

and $x = 2$, or -3.

35. Given $\frac{x}{2} = 22\frac{1}{6} + \frac{\sqrt{x}}{3}$, to find the values of x.

Multiplying by 2, and transposing, $x - \frac{2}{3}\sqrt{x} = \frac{133}{3}$;

and completing the square,

$$x - \frac{2}{3}\sqrt{x} + \frac{1}{9} = \frac{133}{3} + \frac{1}{9} = \frac{400}{9};$$

$$\therefore \text{ extracting the root, } \sqrt{x} - \frac{1}{3} = \pm\frac{20}{3},$$

$$\text{and } \sqrt{x} = 7, \text{ or } -\frac{19}{3};$$

$$\therefore\ x = 49, \text{ or } \frac{361}{9}.$$

36. Given $\dfrac{\frac{3\sqrt{x}}{5} - 2}{x - 5} - \dfrac{1}{20} = 0$, to find the values of x.

Clearing the equation of fractions,

$$12\sqrt{x} - 40 - x + 5 = 0;$$

$$\therefore\ (17.\ \text{Cor. } 1.)\ x - 12\sqrt{x} = -35,$$

and completing the square, $x - 12\sqrt{x} + 36 = 36 - 35 = 1$;

$$\therefore \text{ extracting the root, } \sqrt{x} - 6 = \pm 1;$$

$$\therefore\ \sqrt{x} = 7, \text{ or } 5,$$

$$\text{and } x = 49, \text{ or } 25.$$

37. Given $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$, to find the values of x.

Completing the square, $x^{\frac{6}{5}} + x^{\frac{3}{5}} + \frac{1}{4} = 756 + \frac{1}{4} = \frac{3025}{4}$;

and extracting the root, $x^{\frac{3}{5}} + \frac{1}{2} = \pm\frac{55}{2}$;

$$\therefore\ x^{\frac{3}{5}} = 27, \text{ or } -28,$$

$$\text{and } x^{\frac{1}{5}} = 3, \text{ or } \sqrt[3]{(-28)};$$

$$\therefore\ x = 243, \text{ or } (-28)^{\frac{5}{3}}.$$

38. Given $x^3 - x^{\frac{3}{2}} = 56$, to find the values of x.

Completing the square, $x^3 - x^{\frac{3}{2}} + \frac{1}{4} = 56 + \frac{1}{4} = \frac{225}{4}$;

$\therefore$ extracting the root, $x^{\frac{3}{2}} - \frac{1}{2} = \pm \frac{15}{2}$;

and $x^{\frac{3}{2}} = 8$, or -7;

$\therefore x^{\frac{1}{2}} = 2$, or $\sqrt[3]{(-7)}$,

and $x = 4$, or $(-7)^{\frac{2}{3}}$.

39. Given $3x^{\frac{5}{3}} + x^{\frac{5}{6}} = 3104$, to find the values of x.

Dividing by 3, $x^{\frac{5}{3}} + \frac{1}{3}x^{\frac{5}{6}} = \frac{3104}{3}$;

and completing the square,

$$x^{\frac{5}{3}} + \frac{1}{3}x^{\frac{5}{6}} + \frac{1}{36} = \frac{3104}{3} + \frac{1}{36} = \frac{37249}{36};$$

$\therefore$ extracting the root, $x^{\frac{5}{6}} + \frac{1}{6} = \pm \frac{193}{6}$;

whence $x^{\frac{5}{6}} = 32$, or $-\frac{97}{3}$,

and $x^{\frac{1}{6}} = 2$, or $\left(\frac{-97}{3}\right)^{\frac{1}{5}}$;

$\therefore x = 64$, or $\left(\frac{-97}{3}\right)^{\frac{6}{5}}$.

40. Given $ax^{\frac{2}{3}} + bx^{\frac{1}{3}} = c$, to find the values of x.

Dividing by a, $x^{\frac{2}{3}} + \frac{b}{a}x^{\frac{1}{3}} = \frac{c}{a}$;

and completing the square,

$$x^{\frac{2}{3}} + \frac{b}{a}x^{\frac{1}{3}} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2 + 4ac}{4a^2};$$

$\therefore$ extracting the root, $x^{\frac{1}{3}} + \frac{b}{2a} = \frac{\pm\sqrt{(b^2 + 4ac)}}{2a}$,

and $x^{\frac{1}{3}} = \frac{\pm\sqrt{(b^2 + 4ac)} - b}{2a}$;

$$\text{whence } x = \left(\frac{\pm\sqrt{(b^2 + 4ac)} - b}{2a}\right)^{\frac{1}{2}}.$$

41. Given $3x^{\frac{4}{3}} - \frac{5x^{\frac{8}{3}}}{2} = -592$, to find the values of x.

$$(17.\ \text{Cor. 1.})\ \frac{5x^{\frac{8}{3}}}{2} - 3x^{\frac{4}{3}} = 592;$$

$$\text{and therefore } x^{\frac{8}{3}} - \frac{6}{5}x^{\frac{4}{3}} = \frac{1184}{5}.$$

Completing the square,

$$x^{\frac{8}{3}} - \frac{6}{5}x^{\frac{4}{3}} + \frac{9}{25} = \frac{1184}{5} + \frac{9}{25} = \frac{5929}{25};$$

$$\therefore \text{ extracting the root, } x^{\frac{4}{3}} - \frac{3}{5} = \pm\frac{77}{5};$$

$$\therefore x^{\frac{4}{3}} = 16, \text{ or } -\frac{74}{5},$$

$$\text{and } x = 8, \text{ or } \left(-\frac{74}{5}\right)^{\frac{3}{4}}.$$

42. Given $x^n - 2ax^{\frac{n}{2}} = b$, to find the values of x.

$$\text{Completing the square, } x^n - 2ax^{\frac{n}{2}} + a^2 = a^2 + b;$$

$$\therefore \text{ extracting the root, } x^{\frac{n}{2}} - a = \pm\sqrt{(a^2 + b)},$$

$$\text{and } x^{\frac{n}{2}} = a \pm \sqrt{(a^2 + b)};$$

$$\therefore x = \{a \pm \sqrt{(a^2 + b)}\}^{\frac{2}{n}}.$$

43. Given $a^2x^2 - bx = c$, to find the values of x.

$$\text{The equation is the same as } a^2x^2 - \frac{b}{a}\cdot ax = c;$$

$$\therefore \text{ completing the square, } a^2x^2 - \frac{b}{a}\cdot ax + \frac{b^2}{4a^2} = c + \frac{b^2}{4a^2};$$

$$\text{extracting the root, } ax - \frac{b}{2a} = \pm \sqrt{\left(c + \frac{b^2}{4a^2}\right)};$$

$$\therefore\ ax = \frac{b \pm \sqrt{(4a^2c + b^2)}}{2a},$$

$$\text{and } x = \frac{b \pm \sqrt{(4a^2c + b^2)}}{2a^2}.$$

44. Given $\sqrt{(2x + 1)} + 2\sqrt{x} = \frac{21}{\sqrt{(2x + 1)}}$, to find the values of x.

$$(18.\ \text{Cor. } 1.)\ 2x + 1 + 2\sqrt{(2x^2 + x)} = 21;$$

$$\therefore \text{ by transposition, } 2\sqrt{(2x^2 + x)} = 20 - 2x;$$

$$\text{and therefore } \sqrt{(2x^2 + x)} = 10 - x;$$

$$\therefore \text{ squaring both sides, } 2x^2 + x = 100 - 20x + x^2;$$

$$\text{and transposing, } x^2 + 21x = 100;$$

completing the square,

$$x^2 + 21x + \left(\frac{21}{2}\right)^2 = 100 + \frac{441}{4} = \frac{841}{4};$$

$$\therefore \text{ extracting the root, } x + \frac{21}{2} = \pm \frac{29}{2};$$

$$\therefore\ x = 4, \text{ or } -25.$$

45. Given $2\sqrt{(x - a)} + 3\sqrt{(2x)} = \frac{7a + 5x}{\sqrt{(x - a)}}$, to find the values of x.

$$(18.\ \text{Cor. } 1.)\ 2x - 2a + 3\sqrt{(2x^2 - 2ax)} = 7a + 5x;$$

$$\text{by transposition, } 3\sqrt{(2x^2 - 2ax)} = 9a + 3x;$$

$$\therefore\ \sqrt{(2x^2 - 2ax)} = 3a + x;$$

$$\text{squaring both sides, } 2x^2 - 2ax = 9a^2 + 6ax + x^2;$$

$$\text{by transposition, } x^2 - 8ax = 9a^2;$$

$$\text{completing the square, } x^2 - 8ax + 16a^2 = 25a^2;$$

$$\text{extracting the root, } x - 4a = \pm 5a;$$

$$\therefore\ x = 9a, \text{ or } -a.$$

46. Given

$$\sqrt{(x+60)}+\sqrt{(x^2+9)}=\frac{2\sqrt{(x^3+60x^2+9x+540)}+89}{\sqrt{(x+60)}+\sqrt{(x^2+9)}},$$

to find the values of x.

Clearing the equation of fractions,

$$x+60+x^2+9+2\sqrt{(x^3+60x^2+9x+540)}=$$
$$2\sqrt{(x^3+60x^2+9x+540)}+89;$$

$\therefore$ by transposition, $x^2+x=20$;

and completing the square, $x^2+x+\frac{1}{4}=20+\frac{1}{4}=\frac{81}{4}$;

$\therefore$ extracting the root, $x+\frac{1}{2}=\pm\frac{9}{2}$,

and $x=4$, or -5.

47. Given

$$\frac{123+41\sqrt{x}}{5\sqrt{x}-x}=\frac{20\sqrt{x}+4x}{3-\sqrt{x}}-\frac{2x^2}{(5\sqrt{x}-x)\cdot(3-\sqrt{x})},$$

to find the values of x.

This equation is

$$\frac{41\cdot(3+\sqrt{x})}{5\sqrt{x}-x}=\frac{4\cdot(5\sqrt{x}+x)}{3-\sqrt{x}}-\frac{2x^2}{(5\sqrt{x}-x)\cdot(3-\sqrt{x})};$$

and therefore, clearing it of fractions,

$$41\cdot(9-x)=4\cdot(25x-x^2)-2x^2,$$
$$\text{or } 369-41x=100x-4x^2-2x^2;$$

$\therefore$ by transposition, $6x^2-141x=-369$,

and $x^2-\frac{47}{2}x=-\frac{123}{2}$;

completing the square,

$$x^2-\frac{47}{2}x+\left(\frac{47}{4}\right)^2=\frac{2209}{16}-\frac{123}{2}=\frac{1225}{16};$$

$\therefore$ extracting the root, $x-\frac{47}{4}=\pm\frac{35}{4}$,

and $x=\frac{41}{2}$, or 3.

48. Given $\dfrac{x}{\sqrt{x}+\sqrt{(a-x)}}+\dfrac{x}{\sqrt{x}-\sqrt{(a-x)}}=\dfrac{b}{\sqrt{x}}$, to find the values of x.

Multiplying the equation by

$$\{\sqrt{x}+\sqrt{(a-x)}\}\,.\,\{\sqrt{x}-\sqrt{(a-x)}\};$$

$$x\,.\,\{\sqrt{x}-\sqrt{(a-x)}+\sqrt{x}+\sqrt{(a-x)}\}=\frac{b}{\sqrt{x}}\,.\,(2x-a),$$

$$\text{or } 2x\sqrt{x}=\frac{b}{\sqrt{x}}\,.\,(2x-a);$$

$$\therefore\ 2x^2=2bx-ab,$$

$$\text{and } x^2-bx=-\frac{ab}{2};$$

completing the square,

$$x^2-bx+\frac{b^2}{4}=\frac{b^2}{4}-\frac{ab}{2}=\frac{b^2-2ab}{4};$$

$$\text{extracting the root, } x-\frac{b}{2}=\pm\frac{\sqrt{(b^2-2ab)}}{2};$$

$$\therefore\ x=\frac{b\pm\sqrt{(b^2-2ab)}}{2}.$$

49. Given $\dfrac{x+\sqrt{(x^2-9)}}{x-\sqrt{(x^2-9)}}=(x-2)^2$, to find the values of x.

Multiplying the numerator and the denominator of the fraction by $x+\sqrt{(x^2-9)}$,

$$\frac{\{x+\sqrt{(x^2-9)}\}^2}{9}=(x-2)^2;$$

$$\text{extracting the square root, } \frac{x+\sqrt{(x^2-9)}}{3}=\pm(x-2);$$

$$\text{taking the positive sign, } x+\sqrt{(x^2-9)}=3x-6;$$

$$\text{by transposition, } \sqrt{(x^2-9)}=2x-6,$$

$$\text{and squaring both sides, } x^2-9=4x^2-24x+36;$$

by transposition, $3x^2 - 24x = -45$,

and $x^2 - 8x = -15$;

completing the square, $x^2 - 8x + 16 = 1$;

extracting the root, $x - 4 = \pm 1$;

$\therefore x = 5$, or 3.

But if $\dfrac{x + \sqrt{(x^2 - 9)}}{3} = -(x - 2)$;

$$x + \sqrt{(x^2 - 9)} = -3x + 6,$$

and $\sqrt{(x^2 - 9)} = -4x + 6$;

$\therefore$ squaring both sides, $x^2 - 9 = 16x^2 - 48x + 36$,

and by transposition, $15x^2 - 48x = -45$;

$$\therefore x^2 - \frac{16}{5} . x = -3;$$

and completing the square,

$$x^2 - \frac{16}{5} . x + \frac{64}{25} = \frac{64}{25} - 3 = \frac{-11}{25};$$

extracting the root, $x - \dfrac{8}{5} = \pm \dfrac{\sqrt{(-11)}}{5}$;

$$\therefore x = \frac{8 \pm \sqrt{(\ \ 11)}}{5}.$$

50. Given $x + 5 = \sqrt{(x + 5)} + 6$, to find the values of x.

By transposition, $(x + 5) - \sqrt{(x + 5)} = 6$;

and therefore, completing the square,

$$(x + 5) - \sqrt{(x + 5)} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

extracting the root, $\sqrt{(x + 5)} - \dfrac{1}{2} = \pm \dfrac{5}{2}$;

$\therefore \sqrt{(x + 5)} = 3$, or -2;

and squaring both sides, $x + 5 = 9$, or 4;

whence $x = 4$, or -1.

51. Given $x + 16 - 7\sqrt{(x + 16)} = 10 - 4\sqrt{(x + 16)}$, to find the values of x.

By transposition, $(x + 16) - 3\sqrt{(x + 16)} = 10$;

and completing the square,

$$(x + 16) - 3\sqrt{(x + 16)} + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4};$$

$$\therefore \text{ extracting the root, } \sqrt{(x + 16)} - \frac{3}{2} = \pm \frac{7}{2};$$

$$\text{and } \sqrt{(x + 16)} = 5, \text{ or } -2;$$

$$\text{whence } x + 16 = 25, \text{ or } 4;$$

$$\text{and } x = 9, \text{ or } -12.$$

52. Given $\sqrt{(x + 12)} + \sqrt[4]{(x + 12)} = 6$, to find the values of x.

Completing the square,

$$\sqrt{(x + 12)} + \sqrt[4]{(x + 12)} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\therefore \text{ extracting the root, } \sqrt[4]{(x + 12)} + \frac{1}{2} = \pm \frac{5}{2},$$

$$\text{and } \sqrt[4]{(x + 12)} = 2, \text{ or } -3;$$

$$\text{whence } x + 12 = 16, \text{ or } 81;$$

$$\text{and } x = 4, \text{ or } 69.$$

53. Given $x^2 - 2x + 6\sqrt{(x^2 - 2x + 5)} = 11$, to find the values of x.

Adding 5 to each side of the equation,

$$(x^2 - 2x + 5) + 6\sqrt{(x^2 - 2x + 5)} = 16;$$

$\therefore$ completing the square,

$$(x^2 - 2x + 5) + 6\sqrt{(x^2 - 2x + 5)} + 9 = 16 + 9 = 25;$$

$$\text{and extracting the root, } \sqrt{(x^2 - 2x + 5)} + 3 = \pm 5,$$

$$\text{and } \sqrt{(x^2 - 2x + 5)} = 2, \text{ or } -8;$$

$$\therefore \text{ squaring both sides, } x^2 - 2x + 5 = 4, \text{ or } 64;$$

$$\text{whence } x^2 - 2x + 1 = 0, \text{ or } 60;$$

and extracting the root, $x - 1 = 0$, or $\pm \sqrt{60}$;*

$$\therefore\ x = 1, \text{ or } 1 \pm 2\sqrt{15}.$$

54. Given $2x^2 + 3x - 5\sqrt{(2x^2 + 3x + 9)} + 3 = 0$, to find the values of x.

Adding 6 to each side, $2x^2 + 3x + 9 - 5\sqrt{(2x^2 + 3x + 9)} = 6$; and completing the square,

$$(2x^2 + 3x + 9) - 5\sqrt{(2x^2 + 3x + 9)} + \frac{25}{4} = 6 + \frac{25}{4} = \frac{49}{4};$$

$$\therefore \text{ extracting the root, } \sqrt{(2x^2 + 3x + 9)} - \frac{5}{2} = \pm \frac{7}{2};$$

$$\text{and } \sqrt{(2x^2 + 3x + 9)} = 6, \text{ or } -1,$$

suppose the value to be 6,

$$\text{then } 2x^2 + 3x + 9 = 36,$$

$$\text{and } x^2 + \frac{3}{2}x = \frac{27}{2};$$

$\therefore$ completing the square,

$$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{27}{2} + \frac{9}{16} = \frac{225}{16};$$

$$\text{and extracting the root, } x + \frac{3}{4} = \pm \frac{15}{4},$$

$$\therefore\ x = 3, \text{ or } -\frac{9}{2}.$$

But if -1 be taken, $\sqrt{(2x + 3x + 9)} = -1$;

$$\therefore\ 2x^2 + 3x + 9 = 1,$$

$$\text{and } x^2 + \frac{3}{2}x = -4;$$

completing the square,

$$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{9}{16} - 4 = \frac{-55}{16};$$

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* In this example, if 0 be the value, the two roots of the equation are + 1 and + 1.

$$\text{and extracting the root, } x + \frac{3}{4} = \frac{\pm\sqrt{(-55)}}{4};$$

$$\text{and } x = \frac{-3 \pm \sqrt{(-55)}}{4}.$$

55. Given $\dfrac{\sqrt{(x^2 + x + 6)}}{3} = \dfrac{18 - \left(\frac{4}{3}\sqrt{(x^2 + x + 6)} - 2\right)}{\sqrt{(x^2 + x + 6)}}$,
to find the values of x.

(18. Cor. 1.) $(x^2 + x + 6) = 54 - 4\sqrt{(x^2 + x + 6)} + 6$;

$\therefore$ by transposition, $(x^2 + x + 6) + 4\sqrt{(x^2 + x + 6)} = 60$;

and completing the square,

$$(x^2 + x + 6) + 4\sqrt{(x^2 + x + 6)} + 4 = 64;$$

$$\text{extracting the root, } \sqrt{(x^2 + x + 6)} + 2 = \pm 8,$$

and $\sqrt{(x^2 + x + 6)} = 6$, or -10; suppose the former,

$$\text{then, squaring both sides, } x^2 + x + 6 = 36;$$

$$\text{and by transposition, } x^2 + x = 30;$$

$$\text{completing the square, } x^2 + x + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4};$$

$$\text{extracting the root, } x + \frac{1}{2} = \pm\frac{11}{2},$$

$$\text{and } x = 5, \text{ or } -6.$$

$$\text{But taking } \sqrt{(x^2 + x + 6)} = -10,$$

$$\text{then } x^2 + x + 6 = 100;$$

$$\text{and by transposition, } x^2 + x = 94;$$

$$\text{completing the square, } x^2 + x + \frac{1}{4} = 94 + \frac{1}{4} = \frac{377}{4};$$

$$\text{extracting the root, } x + \frac{1}{2} = \frac{\pm\sqrt{(377)}}{2},$$

$$\text{and } x = \frac{-1 \pm \sqrt{(377)}}{2}.$$

56. Given $\{(x-2)^2 - x\}^2 - (x-2)^2 = 88 - (x-2)$, to find the values of x.

By transposition, $\{(x-2)^2 - x\}^2 - \{(x-2)^2 - x\} = 90$;
and completing the square,

$$\{(x-2)^2 - x\}^2 - \{(x-2)^2 - x\} + \frac{1}{4} = 90 + \frac{1}{4} = \frac{361}{4};$$

extracting the root, $(x-2)^2 - x - \frac{1}{2} = \pm\frac{19}{2}$,

and $(x-2)^2 - x = 10$, or -9;

whence, adding 2 to each side,

$(x-2)^2 - (x-2) = 12$, or -7; supposing the former,
and completing the square,

$$(x-2)^2 - (x-2) + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

extracting the root, $x - 2 - \frac{1}{2} = \pm\frac{7}{2}$;

whence $x = \frac{5}{2} \pm \frac{7}{2} = 6$, or -1;

and in the second case, where $(x-2)^2 - (x-2) = -7$;
completing the square,

$$(x-2)^2 - (x-2) + \frac{1}{4} = \frac{1}{4} - 7 = \frac{-27}{4};$$

extracting the root, $x - 2 - \frac{1}{2} = \frac{\pm\sqrt{(-27)}}{2}$;

$$\therefore\ x = \frac{5 \pm 3\sqrt{(-3)}}{2}.$$

57. Given $(x+6)^2 + 2x^{\frac{1}{2}}.(x+6) = 138 + x^{\frac{1}{2}}$, to find the values of x.

Completing the square,

$$(x+6)^2 + 2x^{\frac{1}{2}}.(x+6) + x = 138 + x + x^{\frac{1}{2}};$$

extracting the root, $x + 6 + x^{\frac{1}{2}} = \pm\sqrt{(138 + x + x^{\frac{1}{2}})}$;
and squaring both sides,

$$(x + x^{\frac{1}{2}})^2 + 12 \cdot (x + x^{\frac{1}{2}}) + 36 = 138 + x + x^{\frac{1}{2}};$$

$\therefore$ by transposition, $(x + x^{\frac{1}{2}})^2 + 11 \cdot (x + x^{\frac{1}{2}}) = 102$;
completing the square,

$$(x + x^{\frac{1}{2}})^2 + 11 \cdot (x + x^{\frac{1}{2}}) + \frac{121}{4} = 102 + \frac{121}{4} = \frac{529}{4};$$

extracting the root, $x + x^{\frac{1}{2}} + \frac{11}{2} = \pm\frac{23}{2}$,

and $x + x^{\frac{1}{2}} = 6$, or -17; supposing the former,

and completing the square, $x + x^{\frac{1}{2}} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

extracting the root, $x^{\frac{1}{2}} + \frac{1}{2} = \pm\frac{5}{2}$,

and $x^{\frac{1}{2}} = 2$, or -3;

$\therefore\ x = 4$, or 9.

But if $x + x^{\frac{1}{2}} = -17$,

completing the square, $x + x^{\frac{1}{2}} + \frac{1}{4} = \frac{1}{4} - 17 = \frac{-67}{4}$;

and extracting the root, $x^{\frac{1}{2}} + \frac{1}{2} = \frac{\pm\sqrt{(-67)}}{2}$,

and $x^{\frac{1}{2}} = \frac{-1 \pm \sqrt{(-67)}}{2}$;

whence $x = \frac{-33 \mp \sqrt{(-67)}}{2}$.

58. Given $x - 1 = 2 + \frac{2}{x^{\frac{1}{2}}}$, to find the values of x.

Since $x - 1 = (x^{\frac{1}{2}} + 1) \times (x^{\frac{1}{2}} - 1)$;

$\therefore\ (x^{\frac{1}{2}} + 1) \cdot (x^{\frac{1}{2}} - 1) = 2 \cdot \frac{x^{\frac{1}{2}} + 1}{x^{\frac{1}{2}}}$;

and therefore, dividing by $x^{\frac{1}{2}} + 1$, $x^{\frac{1}{2}} - 1 = \frac{2}{x^{\frac{1}{2}}}$;

whence $x - x^{\frac{1}{2}} = 2$;

and completing the square, $x - x^{\frac{1}{2}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$;

$\therefore$ extracting the root, $x^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{3}{2}$,

and $x^{\frac{1}{2}} = 2$, or -1;

whence $x = 4$, or 1.

59. Given $x^4 - 2x^3 + x = 132$, to find the values of x.

(15) Adding and subtracting x^2, there results

$$x^4 - 2x^3 + x^2 - (x^2 - x) = 132;$$

and completing the square,

$$(x^2 - x)^2 - (x^2 - x) + \frac{1}{4} = 132 + \frac{1}{4} = \frac{529}{4};$$

extracting the root, $x^2 - x - \frac{1}{2} = \pm \frac{23}{2}$;

whence $x^2 - x = 12$, or -11; supposing the former;

then, completing the square, $x^2 - x + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$;

and extracting the root, $x - \frac{1}{2} = \pm \frac{7}{2}$,

and $x = 4$, or -3.

But if $x^2 - x = -11$,

completing the square, $x^2 - x + \frac{1}{4} = \frac{1}{4} - 11 = -\frac{43}{4}$;

extracting the root, $x - \frac{1}{2} = \frac{\pm\sqrt{(-43)}}{2}$,

and $x = \frac{1 \pm \sqrt{(-43)}}{2}$.

60. Given $(x^2 + 2x) \cdot (x + 4) = 2 - (x + 4)$, to find the values of x.

$$\text{By subtraction, } (x^2 + 2x) \cdot (x + 4) = -(x + 2);$$

$$\therefore \text{ dividing by } x + 2,\ x \cdot (x + 4) = -1;$$

$$\therefore\ x^2 + 4x = -1;$$

$$\text{and completing the square, } x^2 + 4x + 4 = 4 - 1 = 3;$$

$$\text{extracting the root, } x + 2 = \pm\sqrt{3};$$

$$\therefore\ x = -2 \pm \sqrt{3}.$$

61. Given $\dfrac{33\frac{24}{25}}{\sqrt{(5x^2 - x^4)}} + \dfrac{\sqrt{(5 - x^2)}}{25x} = \dfrac{34}{x}$, to find the values of x.

$$\text{Multiplying every term by } 25\sqrt{(5x^2 - x^4)};$$

$$\therefore\ 849 + 5 - x^2 = 850\sqrt{(5 - x^2)};$$

$$\text{and by transposition, } (5 - x^2) - 850\sqrt{(5 - x^2)} = -849;$$

completing the square,

$$(5 - x^2) - 850\sqrt{(5 - x^2)} + (425)^2 = 180625 - 849 = 179776;$$

$$\text{and extracting the root, } \sqrt{(5 - x^2)} - 425 = \pm 424;$$

$$\text{whence } \sqrt{(5 - x^2)} = 849, \text{ or } 1,$$

$$\text{and } 5 - x^2 = 720801, \text{ or } 1,$$

$$\text{and } x^2 = -720796, \text{ or } 4;$$

$$\therefore\ x = \pm\sqrt{(-720796)}, \text{ or } \pm 2.$$

62. Given $\dfrac{a-x}{x} + \dfrac{x}{a-x} = \dfrac{b}{c}$, to find the values of x.

Multiplying every term by $\dfrac{a-x}{x}$;

$$\left(\frac{a-x}{x}\right)^2 + 1 = \frac{b}{c} \cdot \frac{a-x}{x};$$

by transposition, $\left(\frac{a-x}{x}\right)^2 - \frac{b}{c} \cdot \frac{a-x}{x} = -1$;

completing the square,

$$\left(\frac{a-x}{x}\right)^2 - \frac{b}{c} \cdot \frac{a-x}{x} + \frac{b^2}{4c^2} = \frac{b^2}{4c^2} - 1 = \frac{b^2 - 4c^2}{4c^2};$$

extracting the root, $\frac{a-x}{x} - \frac{b}{2c} = \pm \frac{\sqrt{(b^2 - 4c^2)}}{2c}$,

or, $\frac{a}{x} - 1 - \frac{b}{2c} = \pm \frac{\sqrt{(b^2 - 4c^2)}}{2c}$;

by transposition, $\frac{a}{x} = \frac{2c + b \pm \sqrt{(b^2 - 4c^2)}}{2c}$;

$$\therefore\ x = \frac{2ac}{2c + b \pm \sqrt{(b^2 - 4c^2)}}.$$

63. Given $9x + \sqrt{(16x^3 + 36x^2)} = 15x^2 - 4$, to find the values of x.

By transposition, $9x + 4 + 2x\sqrt{(9x + 4)} = 15x^2$;

and completing the square,

$$(9x + 4) + 2x\sqrt{(9x + 4)} + x^2 = 16x^2;$$

and extracting the root, $\sqrt{(9x + 4)} + x = \pm 4x$;

$\therefore\ \sqrt{(9x + 4)} = 3x$, or $-5x$,

and $9x + 4 = 9x^2$, or $25x^2$; supposing the former;

then, by transposition, $9x^2 - 9x = 4$;

and completing the square, $9x^2 - 9x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4}$;

$\therefore$ extracting the root, $3x - \frac{3}{2} = \pm \frac{5}{2}$;

and $3x = 4$, or -1;

whence $x = \frac{4}{3}$, or $-\frac{1}{3}$.

$$\text{But if } 9x + 4 = 25x^2,$$

$$\text{then, by transposition, } 25x^2 - 9x = 4;$$

and completing the square,

$$25x^2 - 9x + \left(\frac{9}{10}\right)^2 = \frac{81}{100} + 4 = \frac{481}{100};$$

$$\text{and extracting the root, } 5x - \frac{9}{10} = \pm\frac{\sqrt{(481)}}{10};$$

$$\therefore\ 5x = \frac{9 \pm \sqrt{(481)}}{10},$$

$$\text{and } x = \frac{9 \pm \sqrt{(481)}}{50}.$$

64. Given $x = \dfrac{12 + 8x^{\frac{1}{2}}}{x - 5}$, to find the values of x.

$$\text{(18. Cor. 1.) } x^2 - 5x = 12 + 8x^{\frac{1}{2}},$$

$$\text{or } x^2 - 4x = 12 + 8x^{\frac{1}{2}} + x;$$

$$\text{and completing the square, } x^2 - 4x + 4 = 16 + 8x^{\frac{1}{2}} + x;$$

extracting the root, $x - 2 = \pm(4 + x^{\frac{1}{2}})$, and first taking the positive value;

$$\text{then, by transposition, } x - x^{\frac{1}{2}} = 6;$$

$$\text{completing the square, } x - x^{\frac{1}{2}} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\text{extracting the root, } x^{\frac{1}{2}} - \frac{1}{2} = \pm\frac{5}{2};$$

$$\therefore\ x^{\frac{1}{2}} = 3, \text{ or } -2,$$

$$\text{and } x = 9, \text{ or } 4.$$

$$\text{But if the negative value be used, } x - 2 = -4 - x^{\frac{1}{2}};$$

$$\therefore\ \text{by transposition, } x + x^{\frac{1}{2}} = -2;$$

$$\text{and completing the square, } x + x^{\frac{1}{2}} + \frac{1}{4} = \frac{1}{4} - 2 = \frac{-7}{4};$$

extracting the root, $x^{\frac{1}{4}} + \frac{1}{2} = \frac{\pm\sqrt{(-7)}}{2}$;

$$\therefore\ x^{\frac{1}{4}} = \frac{-1 \pm \sqrt{(-7)}}{2},$$

$$\text{and } x = \frac{-3 \mp \sqrt{(-7)}}{2}.$$

65. Given $\frac{49x^2}{4} + \frac{48}{x^2} - 49 = 9 + \frac{6}{x}$, to find the values of x.

Adding $\frac{1}{x^2}$ to each side, in order to complete the square,

$$\text{then } \frac{49x^2}{4} - 49 + \frac{49}{x^2} = 9 + \frac{6}{x} + \frac{1}{x^2};$$

extracting the root, $\frac{7x}{2} - \frac{7}{x} = \pm\left(3 + \frac{1}{x}\right)$,

and first taking the positive value;

$\therefore$ by transposition, $\frac{7x}{2} - 3 = \frac{8}{x}$;

and therefore, $x^2 - \frac{6x}{7} = \frac{16}{7}$;

completing the square,

$$x^2 - \frac{6x}{7} + \frac{9}{49} = \frac{16}{7} + \frac{9}{49} = \frac{121}{49};$$

extracting the root, $x - \frac{3}{7} = \pm\frac{11}{7}$,

$$\text{and } x = 2, \text{ or } -\frac{8}{7}.$$

But if the negative value be used, $\frac{7x}{2} - \frac{7}{x} = -3 - \frac{1}{x}$;

$$\therefore \text{ by transposition, } \frac{7x}{2} + 3 = \frac{6}{x},$$

$$\text{and } x^2 + \frac{6x}{7} = \frac{12}{7};$$

$\therefore$ completing the square,

$$x^2 + \frac{6x}{7} + \frac{9}{49} = \frac{12}{7} + \frac{9}{49} = \frac{93}{49};$$

$$\text{and extracting the root, } x + \frac{3}{7} = \frac{\pm\sqrt{(93)}}{7};$$

$$\therefore x = \frac{-3 \pm \sqrt{(93)}}{7}.$$

66. Given $\frac{x^4}{2} + \frac{17x^3}{4} - 17x = 8$, to find the values of x.

$$\text{Multiplying by } 2,\ x^4 + \frac{17x^3}{2} - 34x = 16,$$

$$\therefore \text{ by transposition, } x^4 + \frac{17x^3}{2} = 34x + 16;$$

and completing the square,

$$x^4 + \frac{17x^3}{2} + \left(\frac{17x}{4}\right)^2 = \left(\frac{17x}{4}\right)^2 + 34x + 16;$$

$$\text{extracting the root, } x^2 + \frac{17x}{4} = \pm\left(\frac{17x}{4} + 4\right);$$

first, let the positive value be taken;

then, by transposition, $x^2 = 4$,

and $x = \pm 2$.

But if the negative value be taken,

$$x^2 + \frac{17x}{4} = -\frac{17x}{4} - 4;$$

and by transposition, $x^2 + \frac{17x}{2} = -4$;

$\therefore$ completing the square,

$$x^2 + \frac{17x}{2} + \left(\frac{17}{4}\right)^2 = \frac{289}{16} - 4 = \frac{225}{16};$$

and extracting the root, $x + \frac{17}{4} = \pm\frac{15}{4}$;

$$\therefore x = -8, \text{ or } -\frac{1}{2}.$$

67. Given $27x^2 - \frac{841}{3x^2} + \frac{17}{3} = \frac{232}{3x} - \frac{1}{3x^2} + 5$, to find the values of x.

Multiplying every term by 3,

$$81x^2 - \frac{841}{x^2} + 17 = \frac{232}{x} - \frac{1}{x^2} + 15;$$

$\therefore$ by transposition, $81x^2 + 17 + \frac{1}{x^2} = \frac{841}{x^2} + \frac{232}{x} + 15.$

Adding unity to each side, in order to complete the square;

$$\therefore 81x^2 + 18 + \frac{1}{x^2} = \frac{841}{x^2} + \frac{232}{x} + 16;$$

and extracting the root, $9x + \frac{1}{x} = \pm\left(\frac{29}{x} + 4\right).$

Let the positive value be taken;

then, by transposition, $9x - 4 = \frac{28}{x}$;

and therefore $9x^2 - 4x = 28$;

completing the square, $9x^2 - 4x + \frac{4}{9} = \frac{4}{9} + 28 = \frac{256}{9}$;

extracting the root, $3x - \frac{2}{3} = \pm\frac{16}{3}$;

$$\therefore\ 3x = 6, \text{ or } -\frac{14}{3},$$

$$\text{and } x = 2, \text{ or } -\frac{14}{9}.$$

But if the negative value be taken, $9x^2 + 4x = -30$;

and completing the square,

$$9x^2 + 4x + \frac{4}{9} = \frac{4}{9} - 30 = \frac{-266}{9};$$

$$\text{extracting the root, } 3x + \frac{2}{3} = \frac{\pm\sqrt{(-266)}}{3};$$

$$\therefore\ 3x = \frac{-2 \pm \sqrt{(-266)}}{3}$$

$$\text{and } x = \frac{-2 \pm \sqrt{(-266)}}{9}.$$

68. Given $\left(x^2 - \dfrac{a^4}{x^2}\right)^{\frac{1}{2}} + \left(a^2 - \dfrac{a^4}{x^2}\right)^{\frac{1}{2}} = \dfrac{x^2}{a}$, to find the values of x.

By transposition, $\left(x^2 - \dfrac{a^4}{x^2}\right)^{\frac{1}{2}} - \dfrac{x^2}{a} = -\left(a^2 - \dfrac{a^4}{x^2}\right)^{\frac{1}{2}}$;

and squaring both sides,

$$x^2 - \frac{a^4}{x^2} + \frac{x^4}{a^2} - \frac{2x^2}{a} \cdot \left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} = a^2 - \frac{a^4}{x^2};$$

and by transposition, $x^2 - \dfrac{2x^2}{a} \cdot \left(x^2 - \dfrac{a^4}{x^2}\right)^{\frac{1}{2}} + \dfrac{x^4}{a^2} - a^2 = 0$,

$$\text{or } x^2 - \frac{2x}{a} \cdot (x^4 - a^4)^{\frac{1}{2}} + \frac{x^4 - a^4}{a^2} = 0;$$

$$\text{extracting the root, } x - \frac{\sqrt{(x^4 - a^4)}}{a} = 0;$$

$\therefore$ by transposition, $x = \dfrac{\sqrt{(x^4 - a^4)}}{a}$,

and $ax = \sqrt{(x^4 - a^4)}$.

Squaring both sides, $a^2x^2 = x^4 - a^4$;

$\therefore$ by transposition, $x^4 - a^2x^2 = a^4$;

and completing the square,

$$x^4 - a^2x^2 + \frac{a^4}{4} = a^4 + \frac{a^4}{4} = \frac{5a^4}{4};$$

extracting the root, $x^2 - \dfrac{a^2}{2} = \dfrac{\pm\sqrt{5}\,.\,a^2}{2}$;

$$\therefore\ x^2 = \frac{1 \pm \sqrt{5}}{2}\,.\,a^2,$$

and $x = \pm a\,.\sqrt{\left(\dfrac{1 \pm \sqrt{5}}{2}\right)}$.

SECTION V.

Solution of Adfected Quadratics, involving two unknown Quantities.

(30.) If the equations involve two unknown quantities, they may, by the preceding rules, be reduced to one containing only one of the unknown quantities, the values of which may be found by Art. 29; whence, by substitution, the values of the other may also be determined. In many cases, however, it may be convenient to solve the equations first, considering one of the quantities as known; when the rules for exterminating unknown quantities (23) may be more easily applied.

Examples.

1. Given $\left.\begin{array}{l} x - y = 15 \\ \text{and } \frac{xy}{2} = y^2 \end{array}\right\}$, to find the values of x and y.

From the second equation, $x = 2y^2$;

Substituting this in the first, $2y^2 - y = 15$;

$$\therefore\ y^2 - \frac{1}{2}y = \frac{15}{2};$$

and completing the square,

$$y^2 - \frac{1}{2}y + \frac{1}{16} = \frac{15}{2} + \frac{1}{16} = \frac{121}{16};$$

$$\therefore \text{ extracting the root, } y - \frac{1}{4} = \pm\frac{11}{4},$$

$$\text{and } y = 3, \text{ or } -\frac{5}{2};$$

$$\text{whence } x = 2y^2 = 18, \text{ or } \frac{25}{2}.$$

2. Given $\left.\begin{array}{l}\dfrac{10x+y}{xy} = 3 \\ \text{and } 9y - 9x = 18\end{array}\right\}$, to find the values of x and y.

From the second equation, $y - x = 2$;

and therefore $y = x + 2$;

but from the first, $10x + y = 3xy$.

Substituting in this the value of y found above,

$$10x + x + 2 = 3x \,.\, (x + 2),$$

$$\text{or } 11x + 2 = 3x^2 + 6x;$$

$$\therefore \text{ by transposition, } 3x^2 - 5x = 2,$$

$$\text{and (18) } x^2 - \frac{5}{3}x = \frac{2}{3};$$

$\therefore$ completing the square,

$$x^2 - \frac{5}{3}\,.\,x + \frac{25}{36} = \frac{25}{36} + \frac{2}{3} = \frac{49}{36};$$

$$\therefore \text{ extracting the root, } x - \frac{5}{6} = \pm\frac{7}{6},$$

$$\text{and } x = 2, \text{ or } -\frac{1}{3};$$

$$\text{whence } y = x + 2 = 4, \text{ or } \frac{5}{3}.$$

3. Given $\left.\begin{array}{l}x + y : x - y :: 13 : 5 \\ \text{and } y^2 + x = 25\end{array}\right\}$, to find the values of x and y.

$$(Alg.\ 179,\ 6.)\ 2x : 2y :: 18 : 8;$$

$$\therefore (Alg.\ 179,\ 7.)\ x : y :: 9 : 4;$$

$$\therefore (21)\ 4x = 9y, \text{ and } x = \frac{9y}{4}.$$

Substituting this value in the second equation,

$$y^2 + \frac{9y}{4} = 25;$$

completing the square,

$$y^2 + \frac{9y}{4} + \frac{81}{64} = \frac{81}{64} + 25 = \frac{1681}{64};$$

$$\therefore \text{ extracting the root, } y + \frac{9}{8} = \pm \frac{41}{8};$$

$$\text{whence } y = 4, \text{ or } -\frac{25}{4};$$

$$\therefore x = \frac{9y}{4} = 9, \text{ or } -\frac{225}{16}.$$

4. Given $\left.\begin{array}{l} 4xy = 96 - x^2y^2 \\ \text{and } x + y = 6 \end{array}\right\}$, to find the values of x and y.

From the first equation, $x^2y^2 + 4xy = 96$;

$\therefore$ completing the square, $x^2y^2 + 4xy + 4 = 100$;

and extracting the root, $xy + 2 = \pm 10$;

$$\therefore xy = 8, \text{ or } -12.$$

Now, squaring the second equation,

$$x^2 + 2xy + y^2 = 36;$$

$$\text{but } 4xy = 32, \text{ or } -48;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 4, \text{ or } 84;$$

$$\text{whence, extracting the root, } x - y = \pm 2 \text{ or } \pm\sqrt{84};$$

$$\text{but } x + y = 6;$$

$$\therefore \text{ by addition, } 2x = 8, \text{ or } 4, \text{ or } 6 \pm 2\sqrt{21};$$

$$\text{whence } x = 4, \text{ or } 2, \text{ or } 3 \pm \sqrt{21};$$

$$\text{and by subtraction, } 2y = 4, \text{ or } 8, \text{ or } 6 \mp 2\sqrt{21};$$

$$\therefore y = 2, \text{ or } 4, \text{ or } 3 \mp \sqrt{21}.$$

5. Given $\left.\begin{array}{l} x^n + y^n = 2a^n \\ \text{and} \quad xy = c^2 \end{array}\right\}$, to find the values of x and y.

From the second equation, $y = \dfrac{c^2}{x}$;

and substituting this value in the first equation,

$$x^n + \frac{c^{2n}}{x^n} = 2a^n;$$

$$\therefore x^{2n} + c^{2n} = 2a^n x^n;$$

$$\text{by transposition, } x^{2n} - 2a^n x^n = -c^{2n};$$

$$\text{completing the square, } x^{2n} - 2a^n x^n + a^{2n} = a^{2n} - c^{2n};$$

$$\therefore \text{ extracting the square root, } x^n - a^n = \pm\sqrt{(a^{2n} - c^{2n})},$$

$$\text{and } x^n = a^n \pm \sqrt{(a^{2n} - c^{2n})};$$

$$\therefore x = \left\{a^n \pm \sqrt{(a^{2n} - c^{2n})}\right\}^{\frac{1}{n}},$$

$$\text{and } y = \frac{c^2}{x} = \frac{c^2}{\{a^n \pm \sqrt{(a^{2n} - c^{2n})}\}^{\frac{1}{n}}}.$$

6. Given $\left.\begin{array}{l} x^2 + x + y = 18 - y^2 \\ \text{and } xy = 6 \end{array}\right\}$, to find the values of x and y.

$$\text{By transposition, } x^2 + y^2 + x + y = 18;$$

$$\text{and from the second equation, } \quad 2xy \quad = 12;$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 + x + y = 30;$$

and completing the square,

$$(x + y)^2 + (x + y) + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4};$$

$$\therefore \text{ extracting the root, } x + y + \frac{1}{2} = \pm \frac{11}{2},$$

$$\text{and } x + y = 5, \text{ or } - 6;$$

whence, from the first equation, $x^2 + y^2 = 13$, or 24;

$$\text{but } 2xy = 12;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1, \text{ or } 12;$$

$$\therefore x - y = \pm 1, \text{ or } \pm 2\sqrt{3}.$$

$$\text{Now } x + y = 5, \text{ or } - 6;$$

$$\therefore \text{ by addition, } 2x = 6, \text{ or } 4, \text{ or } - 6 \pm 2\sqrt{3};$$

$$\therefore x = 3, \text{ or } 2, \text{ or } - 3 \pm \sqrt{3};$$

$$\text{and by subtraction, } 2y = 4, \text{ or } 6, \text{ or } - 6 \mp 2\sqrt{3};$$

$$\therefore y = 2, \text{ or } 3, \text{ or } - 3 \mp \sqrt{3}.$$

7. Given $\left.\begin{array}{l} x^2 + 2xy + y^2 + 2y = 120 - 2x \\ \text{and } xy - y^2 = 8 \end{array}\right\}$, to find the values of x and y.

$$\text{By transposition, } (x + y)^2 + 2 \cdot (x + y) = 120;$$

$$\therefore \text{ completing the square, } (x + y)^2 + 2 \cdot (x + y) + 1 = 121;$$

$$\therefore \text{ extracting the root, } (x + y) + 1 = \pm 11,$$

and $x + y = 10$, or $- 12$; and first let $x + y = 10$;

$$\text{from the second equation, } x - y = \frac{8}{y};$$

$$\therefore \text{ by subtraction, } 2y = 10 - \frac{8}{y};$$

$$\therefore y^2 = 5y - 4;$$

$$\text{and by transposition, } y^2 - 5y = - 4;$$

$$\text{completing the square, } y^2 - 5y + \frac{25}{4} = \frac{25}{4} - 4 = \frac{9}{4};$$

$$\text{and extracting the root, } y - \frac{5}{2} = \pm \frac{3}{2};$$

$$\therefore\ y = 4, \text{ or } 1;$$

$$\text{and } x = 10 - y = 6, \text{ or } 9.$$

$$\text{But if } x + y = -12,$$

$$\text{and } x - y = \frac{8}{y};$$

$$\text{then } 2y = -12 - \frac{8}{y},$$

$$\text{and } y^2 + 6y = -4;$$

$$\therefore \text{ completing the square, } y^2 + 6y + 9 = 9 - 4 = 5;$$

$$\text{extracting the root, } y + 3 = \pm\sqrt{5};$$

$$\therefore\ y = -3 \pm \sqrt{5},$$

$$\text{and } x = -12 - y = -9 \mp \sqrt{5}.$$

8. Given $\left.\begin{array}{r} x^2 + y^2 - x - y = 78 \\ \text{and } xy + x + y = 39 \end{array}\right\}$, to find the values of x and y.

$$\text{Since } x^2 + y^2 - (x + y) = 78;$$

$$\text{and from the second, } 2xy + 2 \,.\, (x + y) = 78;$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 + x + y = 156;$$

and completing the square,

$$(x + y)^2 + (x + y) + \frac{1}{4} = 156 + \frac{1}{4} = \frac{625}{4};$$

$$\text{extracting the root, } x + y + \frac{1}{2} = \pm\frac{25}{2};$$

$$\therefore\ x + y = 12, \text{ or } -13; \text{ supposing the former;}$$

$$\text{then } xy = 39 - (x + y) = 39 - 12 = 27,$$

$$\text{and } x^2 + y^2 = 78 + (x + y) = 78 + 12 = 90;$$

$$\text{but } 2xy = \qquad 54;$$

$$\therefore \text{ by subtraction, } \quad x^2 - 2xy + y^2 = 36;$$

and extracting the root, $x - y = \pm 6$;

$$\begin{aligned}
\text{but } x + y &= 12; \\
\therefore \text{ by addition, } 2x &= 18, \text{ or } 6, \\
\text{and } x &= 9, \text{ or } 3; \\
\text{and by subtraction, } 2y &= 6, \text{ or } 18, \\
\text{and } y &= 3, \text{ or } 9.
\end{aligned}$$

But if $x + y = -13$,

$$\begin{aligned}
\text{then } xy &= 39 + 13 = 52, \\
\text{and } x^2 + y^2 &= 78 - 13 = 65; \\
\text{but } 2xy &= 104;
\end{aligned}$$

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = -39$;

and extracting the root, $x - y = \pm\sqrt{(-39)}$;

but $x + y = -13$;

$\therefore$ by addition, $2x = -13 \pm \sqrt{(-39)}$;

$$\text{and } x = \frac{-13 \pm \sqrt{(-39)}}{2};$$

but by subtraction, $2y = -13 \mp \sqrt{(-39)}$;

$$\therefore y = \frac{-13 \mp \sqrt{(-39)}}{2};$$

9. Given $\left.\begin{array}{l} x^2y^4 - 7xy^2 - 945 = 765 \\ \text{and } xy - y = 12 \end{array}\right\}$, to find the values of x and y.

From the first equation, by transposition,

$$x^2y^4 - 7xy^2 = 1710;$$

and completing the square,

$$x^2y^4 - 7xy^2 + \frac{49}{4} = 1710 + \frac{49}{4} = \frac{6889}{4};$$

$$\text{extracting the root, } xy^2 - \frac{7}{2} = \pm\frac{83}{2};$$

$$\therefore xy^2 = 45, \text{ or } -38.$$

Multiplying the second equation by y, $xy^2 - y^2 = 12y$.

Substituting in this the value of xy^2 found above,

$$45 - y^2 = 12y, \text{ in one case;}$$

$$\text{and by transposition, } y^2 + 12y = 45;$$

$$\text{completing the square, } y^2 + 12y + 36 = 45 + 36 = 81;$$

$$\text{extracting the root, } y + 6 = \pm 9,$$

$$\text{and } y = 3, \text{ or } -15;$$

$$\text{whence } x = \frac{45}{y^2} = 5, \text{ or } \frac{1}{5};$$

$$\text{and in the other case, } -38 - y^2 = 12y;$$

$$\text{whence } y^2 + 12y = -38;$$

$$\text{completing the square, } y^2 + 12y + 36 = 36 - 38 = -2;$$

$$\text{extracting the root, } y + 6 = \pm\sqrt{-2};$$

$$\therefore\ y = -6 \pm \sqrt{-2},$$

$$\text{and } x = \frac{-38}{y^2} = \frac{-38}{34 \mp 12\sqrt{-2}} = \frac{-19}{17 \mp 6\sqrt{-2}}.$$

10. Given $\left.\begin{array}{l} x - 2\sqrt{xy} + y - \sqrt{x} + \sqrt{y} = 0 \\ \text{and } \sqrt{x} + \sqrt{y} = 5 \end{array}\right\}$, to find the values of x and y.

Completing the square in the first equation,

$$(\sqrt{x} - \sqrt{y})^2 - (\sqrt{x} - \sqrt{y}) + \frac{1}{4} = \frac{1}{4};$$

$$\text{and extracting the root, } \sqrt{x} - \sqrt{y} - \frac{1}{2} = \pm\frac{1}{2};$$

$$\sqrt{x} - \sqrt{y} = 1, \text{ or } 0;$$

$$\text{but from the second equation, } \sqrt{x} + \sqrt{y} = 5;$$

$$\text{by addition, } 2\sqrt{x} = 6, \text{ or } 5,$$

$$\text{and } \sqrt{x} = 3, \text{ or } \frac{5}{2};$$

$$\therefore\ x = 9, \text{ or } \frac{25}{4};$$

but by subtraction, $2\sqrt{y} = 4$, or 5,

$$\text{or } \sqrt{y} = 2, \text{ or } \frac{5}{2};$$

$$\text{and } y = 4, \text{ or } \frac{25}{4}.$$

11. Given $\left.\begin{array}{l} \dfrac{x^2}{y^2} + \dfrac{4x}{y} = \dfrac{85}{9} \\ \text{and } x - y = 2 \end{array}\right\}$, to find the values of x and y.

Completing the square in the first equation,

$$\frac{x^2}{y^2} + \frac{4x}{y} + 4 = \frac{85}{9} + 4 = \frac{121}{9};$$

$$\text{and extracting the root, } \frac{x}{y} + 2 = \pm\frac{11}{3};$$

$$\therefore\ \frac{x}{y} = \frac{5}{3}, \text{ or } -\frac{17}{3},$$

and $x = \dfrac{5y}{3}$ or $\dfrac{-17y}{3}$; supposing the former;

then, from the second equation, $\dfrac{5y}{3} - y = 2$,

$$\text{or } 2y = 6, \text{ and } y = 3;$$

$$\therefore\ x = \frac{5y}{3} = 5.$$

And if the second value be taken, $\dfrac{-17y}{3} - y = 2$,

$$\text{or } -20y = 6; \ \therefore\ y = -\frac{3}{10};$$

$$\text{and } x = \frac{-17y}{3} = \frac{17}{10}.$$

12. Given $\left.\begin{array}{l} \sqrt{\left(\dfrac{3x}{x+y}\right)} + \sqrt{\left(\dfrac{x+y}{3x}\right)} = 2 \\ \text{and } xy - (x+y) = 54 \end{array}\right\}$, to find the values of x and y.

(18. Cor. 1.) and by transposition,

$$3x - 2\sqrt{(3x)} \cdot \sqrt{(x+y)} + (x+y) = 0;$$

$$\therefore \text{ extracting the root, } \sqrt{(3x)} - \sqrt{(x+y)} = 0;$$

$$\text{by transposition, } \sqrt{(3x)} = \sqrt{(x+y)};$$

$$\text{and squaring both sides, } 3x = x + y,$$

$$\text{and } \therefore 2x = y;$$

substituting this value in the second equation, $2x^2 - 3x = 54$,

$$\text{or } x^2 - \frac{3}{2} \cdot x = 27;$$

whence completing the square,

$$x^2 - \frac{3}{2}x + \frac{9}{16} = 27 + \frac{9}{16} = \frac{441}{16};$$

$$\text{extracting the root, } x - \frac{3}{4} = \pm\frac{21}{4},$$

$$\text{and } x = 6, \text{ or } -\frac{9}{2};$$

$$\text{whence } y = 2x = 12, \text{ or } -9.$$

13. Given $\left.\begin{array}{l} x^4 - 2x^2y + y^2 = 49, \\ \text{and } x^4 - 2x^2y^2 + y^4 - x^2 + y^2 = 20 \end{array}\right\}$, to find the values of x and y.

Completing the square in the second equation,

$$(x^2 - y^2)^2 - (x^2 - y^2) + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4};$$

$$\text{extracting the root, } x^2 - y^2 - \frac{1}{2} = \pm\frac{9}{2};$$

$$\therefore x^2 - y^2 = 5, \text{ or } -4;$$

$\left.\begin{array}{l}\text{but extracting the root} \\ \text{of the first equation,}\end{array}\right\}$ $x^2 - y = \pm 7;$

$\therefore$ by subtraction, $y^2 - y = 2$, or -12, or 11, or -3.

Taking the first value, and completing the square,

$$y^2 - y + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4};$$

$$\text{extracting the root, } y - \frac{1}{2} = \pm\frac{3}{2},$$

$$\text{and } y = 2, \text{ or } -1;$$

$$\therefore\ x = \pm\sqrt{(7 + y)} = \pm 3, \text{ or } \pm\sqrt{6}.$$

Taking the second value, $y^2 - y = -12$;

$$\text{completing the square, } y^2 - y + \frac{1}{4} = \frac{1}{4} - 12 = \frac{-47}{4};$$

$$\text{extracting the root, } y - \frac{1}{2} = \frac{\pm\sqrt{(-47)}}{2},$$

$$\text{and } y = \frac{1 \pm \sqrt{(-47)}}{2};$$

$$\text{whence } x = \pm\sqrt{(y - 7)} = \pm\sqrt{\left(\frac{1 \pm \sqrt{(-47)}}{2} - 7\right)}$$

$$= \pm\sqrt{\left(\frac{-13 \pm \sqrt{(-47)}}{2}\right)}.$$

Taking the third value, $y^2 - y = 11$;

$$\text{completing the square, } y^2 - y + \frac{1}{4} = 11 + \frac{1}{4} = \frac{45}{4};$$

$$\text{extracting the root, } y - \frac{1}{2} = \frac{\pm\sqrt{45}}{2};$$

$$\therefore\ y = \frac{1 \pm 3\sqrt{5}}{2};$$

and

$$x = \pm\sqrt{(7+y)} = \pm\sqrt{\left(7 + \frac{1 \pm 3\sqrt{5}}{2}\right)} = \pm\sqrt{\left(\frac{15 \pm 3\sqrt{5}}{2}\right)}.$$

Taking the fourth value, $y^2 - y = -3$;

completing the square, $y^2 - y + \frac{1}{4} = \frac{1}{4} - 3 = \frac{-11}{4}$;

extracting the root, $y - \frac{1}{2} = \frac{\pm\sqrt{(-11)}}{2}$,

and $y = \frac{1 \pm \sqrt{(-11)}}{2}$;

$\therefore\ x = \pm\sqrt{(y-7)}$

$$= \pm\sqrt{\left(\frac{1 \pm \sqrt{(-11)}}{2} - 7\right)} = \pm\sqrt{\left(\frac{-13 \pm \sqrt{(-11)}}{2}\right)}.$$

14. Given $\left.\begin{array}{l} xy + xy^2 = 12 \\ \text{and } y + xy^3 = 18 \end{array}\right\}$, to find the values of x and y.

From the first equation, $x = \frac{12}{y \cdot (1+y)}$,

and from the second, $x = \frac{18}{1+y^3}$;

whence $\frac{12}{y \cdot (1+y)} = \frac{18}{1+y^3}$, and dividing by $\frac{6}{1+y}$,

$$\frac{2}{y} = \frac{3}{y^2 - y + 1};$$

$$\therefore\ 2y^2 - 2y + 2 = 3y,$$

and by transposition, $2y^2 - 5y = -2$,

or $y^2 - \frac{5}{2} \cdot y = -1$;

completing the square, $y^2 - \frac{5}{2} \cdot y + \frac{25}{16} = \frac{25}{16} - 1 = \frac{9}{16}$;

extracting the root, $y - \frac{5}{4} = \pm\frac{3}{4}$,

$$\therefore\ y = 2, \text{ or } \frac{1}{2};$$

hence $x = \frac{12}{y + y^2} = 2$, or 16.

15. Given $\left.\begin{aligned} x - x^{\frac{1}{2}} &= 3 - y \\ \text{and } 4 - x &= y - y^{\frac{1}{2}} \end{aligned}\right\}$, to find the values of x and y.

Adding the two equations together, $4 - x^{\frac{1}{2}} = 3 - y^{\frac{1}{2}}$;

$$\therefore \text{ by transposition, } y^{\frac{1}{2}} = x^{\frac{1}{2}} - 1,$$

$$\text{and } y = (x^{\frac{1}{2}} - 1)^2.$$

Substituting this value in the first equation,

$$\therefore\ x - x^{\frac{1}{2}} = 3 - x + 2x^{\frac{1}{2}} - 1;$$

$$\therefore \text{ by transposition, } 2x - 3x^{\frac{1}{2}} = 2,$$

$$\text{and } x - \frac{3}{2} \cdot x^{\frac{1}{2}} = 1;$$

$$\text{completing the square, } x - \frac{3}{2}x^{\frac{1}{2}} + \frac{9}{16} = 1 + \frac{9}{16} = \frac{25}{16};$$

$$\text{extracting the root, } x^{\frac{1}{2}} - \frac{3}{4} = \pm \frac{5}{4},$$

$$\text{and } x^{\frac{1}{2}} = 2, \text{ or } -\frac{1}{2},$$

$$\therefore\ x = 4, \text{ or } \frac{1}{4},$$

$$\text{and } y = (x^{\frac{1}{2}} - 1)^2 = 1, \text{ or } \frac{9}{4}.$$

16. Given $\left.\begin{aligned} (x^2 + 1) \cdot y &= xy + 126 \\ \text{and } (x^2 + 1) \cdot y &= x^2y^2 - 744 \end{aligned}\right\}$, to find the values of x and y.

Since quantities which are equal to the same, are equal to each other,

$$x^2y^2 - 744 = xy + 126;$$

$$\therefore \text{ by transposition, } x^2y^2 - xy = 870;$$

$$\text{completing the square, } x^2y^2 - xy + \frac{1}{4} = 870 + \frac{1}{4} = \frac{3481}{4};$$

extracting the root, $xy - \frac{1}{2} = \pm \frac{59}{2}$,

and $xy = 30$, or -29; let the former value be taken,

then from the first equation $(x^2 + 1) \cdot \frac{30}{x} = 156$;

$$\therefore x^2 + 1 = \frac{156x}{30} = \frac{26x}{5};$$

and by transposition, $x^2 - \frac{26x}{5} = -1$;

completing the square, $x^2 - \frac{26}{5} \cdot x + \frac{169}{25} = \frac{169}{25} - 1 = \frac{144}{25}$;

extracting the root, $x - \frac{13}{5} = \pm \frac{12}{5}$;

whence $x = 5$, or $\frac{1}{5}$;

and $\therefore y = \frac{30}{x} = 6$, or 150.

In the second case, $(x^2 + 1) \times -\frac{29}{x} = -29 + 126 = 97$,

and $x^2 + 1 = -\frac{97x}{29}$;

by transposition, $x^2 + \frac{97x}{29} = -1$;

completing the square, $x^2 + \frac{97}{29} \cdot x + \left(\frac{97}{58}\right)^2 = \frac{9409}{3364} - 1 = \frac{6045}{3364}$;

extracting the root, $x + \frac{97}{58} = \pm \frac{\sqrt{(6045)}}{58}$;

$$\therefore x = \frac{-97 \pm \sqrt{(6045)}}{58};$$

$$\therefore y = -\frac{29}{x} = \frac{1682}{97 \mp \sqrt{(6045)}}.$$

17. Given $\left.\begin{array}{l} x + y + \sqrt{(x+y)} = 12 \\ \text{and } x^3 + y^3 = 189 \end{array}\right\}$, to find the values of x and y.

Completing the square in the first equation,

$$x + y + \sqrt{(x+y)} + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

extracting the root, $\sqrt{(x+y)} + \frac{1}{2} = \pm \frac{7}{2}$;

$$\therefore \sqrt{(x+y)} = 3, \text{ or } -4,$$

$$\text{and } x + y = 9, \text{ or } 16;$$

$$\therefore x^3 + 3x^2y + 3xy^2 + y^3 = 729, \text{ or } 4096;$$

$$\text{but } x^3 + y^3 = 189;$$

$\therefore$ by subtraction, $3x^2y + 3xy^2 = 540$, or 3907;

$$\therefore (x+y) \cdot xy = 180, \text{ or } \frac{3907}{3};$$

$\therefore 9xy = 180$ in one case, and $16xy = \frac{3907}{3}$ in the other,

whence in the first case $xy = 20$.

$$\text{Now } x^2 + 2xy + y^2 = 81;$$

$$\text{but } 4xy = 80;$$

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 1$;

and extracting the root, $x - y = \pm 1$;

$$\text{but } x + y = 9;$$

$\therefore$ by addition, $2x = 10$, or 8,

$$\text{and } x = 5, \text{ or } 4;$$

but by subtraction, $2y = 8$, or 10;

$$\text{and } y = 4, \text{ or } 5.$$

Now in the second case, $16xy = \frac{3907}{3}$;

$$\therefore 4xy = \frac{3907}{12};$$

$$\text{and since } x^2 + 2xy + y^2 = 256,$$

$$\text{and } 4xy = \frac{3907}{12};$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = -\frac{835}{12};$$

$$\text{and extracting the root, } x - y = \pm\sqrt{\left(\frac{-835}{12}\right)},$$

$$\text{but } x + y = 16;$$

$$\therefore \text{ by addition, } 2x = 16 \pm \sqrt{\left(\frac{-835}{12}\right)},$$

$$\text{and } x = 8 \pm \frac{1}{4}\sqrt{\left(\frac{-835}{3}\right)};$$

$$\text{but by subtraction, } 2y = 16 \mp \sqrt{\left(\frac{-835}{12}\right)},$$

$$\text{and } y = 8 \mp \frac{1}{4}\sqrt{\left(\frac{-835}{3}\right)}.$$

18. Given $\left.\begin{array}{l} x^2 + y^2 + x - y = 132 \\ \text{and } (x^2 + y^2)\,.\,(x - y) = 1220 \end{array}\right\}$, to find the values of x and y.

From the first equation, $x^2 + y^2 = 132 - (x - y)$;

$$\text{and from the second, } x^2 + y^2 = \frac{1220}{x-y};$$

$$\text{whence } 132 - (x - y) = \frac{1220}{x-y};$$

$$\text{and, } \therefore\ 132\,.\,(x - y) - (x - y)^2 = 1220;$$

$$\text{and (17. Cor. 1.) } (x - y)^2 - 132\,.\,(x - y) = -1220;$$

completing the square,

$$(x - y)^2 - 132\,.\,(x - y) + (66)^2 = 4356 - 1220 = 3136;$$

$$\text{extracting the root, } x - y - 66 = \pm 56;$$

and $x - y = 10$, or 122; supposing the former,

$$x^2 + y^2 = 122;$$

$$\therefore\ 2x^2 + 2y^2 = 244;$$

$$\text{but } x^2 - 2xy + y^2 = 100;$$

$$\therefore \text{ by subtraction, } x^2 + 2xy + y^2 = 144,$$

$$\text{and extracting the root, } x + y = \pm 12;$$

$$\text{but } x - y = 10;$$

$$\therefore \text{ by addition, } 2x = 22, \text{ or } - 2,$$

$$\text{and } x = 11, \text{ or } - 1;$$

$$\text{by subtraction, } 2y = 2, \text{ or } - 22;$$

$$\text{and } y = 1, \text{ or } - 11.$$

$$\text{But if } x - y = 122,$$

$$\text{then } x^2 + y^2 = 10,$$

$$\text{and } x^2 - 2xy + y^2 = (122)^2;$$

$$\text{but } 2x^2 + 2y^2 = 20;$$

$$\therefore \text{ by subtraction, } x^2 + 2xy + y^2 = 20 - (122)^2$$

$$\text{and extracting the root, } x + y = \pm \surd \{20 - (122)^2\}$$

$$\text{but } x - y = 122;$$

$$\therefore \text{ by addition, } 2x = 122 \pm 2 \surd \{5 - (61)^2\}$$

$$\text{and } x = 61 \pm \surd(-3716);$$

$$\text{and by subtraction, } 2y = -122 \pm 2\surd(-3716);$$

$$\therefore\ y = -61 \pm \surd(-3716).$$

19. Given $\left.\begin{array}{l} x^{\frac{2}{3}} y^{\frac{1}{2}} = 2y^2 \\ \text{and } 8x^{\frac{1}{3}} - y^{\frac{3}{4}} = 14 \end{array}\right\}$, to find the values of x and y.

From the first equation, $x^{\frac{2}{3}} = 2y^{\frac{3}{2}}$;

$$\text{and } \therefore\ \frac{1}{2} x^{\frac{2}{3}} = y^{\frac{3}{2}};$$

substituting this value in the second equation,

$$8x^{\frac{1}{3}} - \frac{1}{2}x^{\frac{2}{3}} = 14,$$

$$\text{and } 16x^{\frac{1}{3}} - x^{\frac{2}{3}} = 28;$$

$$\text{or (17. Cor. 1.) } x^{\frac{2}{3}} - 16x^{\frac{1}{3}} = -28;$$

$$\text{completing the square, } x^{\frac{2}{3}} - 16x^{\frac{1}{3}} + 64 = 64 - 28 = 36;$$

$$\text{and extracting the root, } x^{\frac{1}{3}} - 8 = \pm 6;$$

$$\therefore\ x^{\frac{1}{3}} = 14, \text{ or } 2,$$

$$\text{and } x = (14)^3 \text{ or } 8;$$

$$\text{but } y^{\frac{1}{2}} = \frac{1}{2}x^{\frac{2}{3}} = 98, \text{ or } 2;$$

$$\therefore\ y = (98)^2, \text{ or } 4.$$

20. Given $\left.\begin{array}{r} x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x \\ \text{and } x^{\frac{1}{2}} + y^{\frac{1}{3}} = x \end{array}\right\}$, to find the values of x and y.

$$\text{Squaring the second equation, } \therefore\ x + 2x^{\frac{1}{2}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = x^2;$$

$$\text{but } x^{\frac{3}{2}} \qquad + y^{\frac{2}{3}} = 3x;$$

$$\therefore \text{ by subtraction, } x - x^{\frac{3}{2}} + 2x^{\frac{1}{2}}y^{\frac{1}{3}} = x^2 - 3x;$$

$$\text{but from the second equation, } y^{\frac{1}{3}} = x - x^{\frac{1}{2}}.$$

Let this value be substituted in the preceding equation,

$$\text{then } x - x^{\frac{3}{2}} + 2x^{\frac{3}{2}} - 2x = x^2 - 3x;$$

$$\text{and by transposition, } 2x = x^2 - x^{\frac{3}{2}};$$

$$\text{and dividing by } x,\ 2 = x - x^{\frac{1}{2}};$$

$$\text{completing the square, } x - x^{\frac{1}{2}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4};$$

$$\text{extracting the root, } x^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{3}{2},$$

$$\text{and } x^{\frac{1}{2}} = 2, \text{ or } -1;$$

$$\therefore x = 4, \text{ or } 1,$$

$$\text{and } y^{\frac{1}{3}} = x - x^{\frac{1}{2}} = 2;$$

$$\therefore y = 8.$$

21. Given $\left.\begin{array}{r} x + x^{\frac{1}{2}} = \dfrac{y^2 + y + 2}{x^{\frac{1}{2}}} + 4 \\ \text{and } y + xy = y^2 + 4y \end{array}\right\}$, to find the values of x and y.

From the first equation, $x^{\frac{3}{2}} + x - 4x^{\frac{1}{2}} = y^2 + y + 2$,

and from the second, $x = y + 3$.

Substituting this value for x in the former,

$$x^{\frac{3}{2}} + y + 3 - 4x^{\frac{1}{2}} = y^2 + y + 2,$$

and by transposition, $x^{\frac{3}{2}} - 4x^{\frac{1}{2}} = y^2 - 1$.

But since $x = y + 3$; $\therefore x - 4 = y - 1$,

by which equation let the preceding one be divided;

$$\therefore x^{\frac{1}{2}} = y + 1;$$

squaring both sides of this equation, $x = y^2 + 2y + 1$.

Equating therefore the two values of x.

$$y^2 + 2y + 1 = y + 3;$$

$\therefore$ by transposition, $y^2 + y = 2$;

completing the square, $y^2 + y + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$;

extracting the root, $y + \frac{1}{2} = \pm\frac{3}{2}$;

$$\therefore y = 1, \text{ or } -2;$$

whence $x^{\frac{1}{2}} = y + 1 = 2$, or -1.

and therefore $x = 4$, or 1.

22. Given $\left.\begin{array}{r} \dfrac{x^2}{y^2} + \dfrac{y}{x} + \dfrac{x}{y} = 6\dfrac{3}{4} - \dfrac{y^2}{x^2} \\ \text{and } x - y = 2 \end{array}\right\}$, to find the values of x and y.

By transposition, $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} = \frac{27}{4}$;

$\therefore$ adding 2 to each side, $\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} = \frac{35}{4}$;

completing the square, $\left(\frac{x}{y} + \frac{y}{x}\right)^2 + \left(\frac{x}{y} + \frac{y}{x}\right) + \frac{1}{4} = \frac{36}{4}$;

extracting the root, $\frac{x}{y} + \frac{y}{x} + \frac{1}{2} = \pm \frac{6}{2}$;

$$\therefore \frac{x}{y} + \frac{y}{x} = \frac{5}{2}, \text{ or } -\frac{7}{2};$$

$$\therefore x^2 + y^2 = \frac{5xy}{2}, \text{ or } -\frac{7xy}{2};$$

now from the second equation squared,

$$x^2 + y^2 = 2xy + 4;$$

$$\therefore 2xy + 4 = \frac{5xy}{2}, \text{ or } -\frac{7xy}{2};$$

whence by multiplication and transposition,

$$xy = 8, \text{ or } -\frac{8}{11};$$

and since $x^2 - 2xy + y^2 = 4$,

and $4xy = 32$, or $-\frac{32}{11}$;

$\therefore$ by addition, $x^2 + 2xy + y^2 = 36$, or $\frac{12}{11}$;

and extracting the root, $x + y = \pm 6$, or $\pm \frac{2\sqrt{3}}{\sqrt{11}}$;

but $x - y = 2$;

$\therefore$ by addition, $2x = 8$, or -4, or $2 \pm \frac{2\sqrt{3}}{\sqrt{11}}$,

and $x = 4$, or -2, or $1 \pm \frac{\sqrt{3}}{\sqrt{11}}$;

$\therefore$ by subtraction, $2y = 4$, or -8, or $-2 \pm \dfrac{2\sqrt{3}}{\sqrt{11}}$,

$$\therefore\ y = 2, \text{ or } -4, \text{ or } -1 \pm \frac{\sqrt{3}}{\sqrt{11}}.$$

23. Given $2x + y = 26 - 7\sqrt{(2x + y + 4)}$ and $\dfrac{2x + \sqrt{y}}{2x - \sqrt{y}} = \dfrac{16}{15} + \dfrac{2x - \sqrt{y}}{2x + \sqrt{y}}$, to find the values of x and y.

Adding 4 to each side of the first equation,

$$2x + y + 4 = 30 - 7\sqrt{(2x + y + 4)};$$

$\therefore$ by transposition, $2x + y + 4 + 7\sqrt{(2x + y + 4)} = 30$;

completing the square,

$$(2x + y + 4) + 7\sqrt{(2x + y + 4)} + \frac{49}{4} = 30 + \frac{49}{4} = \frac{169}{4};$$

extracting the root, $\sqrt{(2x + y + 4)} + \dfrac{7}{2} = \pm \dfrac{13}{2}$;

$$\therefore\ \sqrt{(2x + y + 4)} = 3, \text{ or } -10,$$

$$\text{and } 2x + y + 4 = 9, \text{ or } 100;$$

$$\therefore\ 2x + y = 5, \text{ or } 96.$$

Multiplying every term of the second equation by $\dfrac{2x + \sqrt{y}}{2x - \sqrt{y}}$,

$$\left(\frac{2x + \sqrt{y}}{2x - \sqrt{y}}\right)^2 = \frac{16}{15} \cdot \frac{2x + \sqrt{y}}{2x - \sqrt{y}} + 1;$$

$\therefore$ by transposition, $\left(\dfrac{2x + \sqrt{y}}{2x - \sqrt{y}}\right)^2 - \dfrac{16}{15} \cdot \dfrac{2x + \sqrt{y}}{2x - \sqrt{y}} = 1$;

completing the square,

$$\left(\frac{2x + \sqrt{y}}{2x - \sqrt{y}}\right)^2 - \frac{16}{15} \cdot \frac{2x + \sqrt{y}}{2x - \sqrt{y}} + \frac{64}{225} = 1 + \frac{64}{225} = \frac{289}{225};$$

extracting the root, $\dfrac{2x + \sqrt{y}}{2x - \sqrt{y}} - \dfrac{8}{15} = \pm \dfrac{17}{15}$;

$$\therefore\ \frac{2x + \sqrt{y}}{2x - \sqrt{y}} = \frac{5}{3}, \text{ or } -\frac{3}{5}.$$

Let the former value be taken, then

$$6x + 3\sqrt{y} = 10x - 5\sqrt{y},$$

and by transposition, $8\sqrt{y} = 4x$,

and $2\sqrt{y} = x$;

but if the second value be taken, $\dfrac{2x + \sqrt{y}}{2x - \sqrt{y}} = -\dfrac{3}{5}$;

$$\therefore\ 10x + 5\sqrt{y} = -6x + 3\sqrt{y},$$

and $16x = -2\sqrt{y}$;

or $8x = -\sqrt{y}$.

Now $2x + y = 5$, or 96; supposing the former,

and taking the first value of $2x = 4\sqrt{y}$,

$$y + 4\sqrt{y} = 5;$$

completing the square, $y + 4\sqrt{y} + 4 = 9$;

extracting the root, $\sqrt{y} + 2 = \pm 3$;

$$\therefore\ \sqrt{y} = 1, \text{ or } -5,$$

and $y = 1$, or 25;

but $x = 2\sqrt{y} = 2$, or -10.

Again, taking the value, $2x = -\dfrac{1}{4}\sqrt{y}$,

$$y - \frac{1}{4}\sqrt{y} = 5;$$

completing the square,

$$y - \frac{1}{4}\sqrt{y} + \frac{1}{64} = 5 + \frac{1}{64} = \frac{321}{64};$$

$\therefore$ extracting the root, $\sqrt{y} - \dfrac{1}{8} = \dfrac{\pm\sqrt{(321)}}{8}$,

and $\sqrt{y} = \dfrac{1 \pm \sqrt{(321)}}{8}$;

$$\therefore\ y = \frac{161 \pm \sqrt{(321)}}{32},$$

and $x = -\dfrac{1}{8}\sqrt{y} = \dfrac{-1 \mp \sqrt{(321)}}{64}$.

Now taking the equation $2x + y = 96$, and the first value $2x = 4\sqrt{y}$;

$$\text{then } y + 4\sqrt{y} = 96;$$

$$\text{completing the square, } y + 4\sqrt{y} + 4 = 100,$$

$$\text{and } \sqrt{y} + 2 = \pm 10;$$

$$\therefore \sqrt{y} = 8, \text{ or } -12;$$

$$\text{and } \therefore y = 64, \text{ or } 144;$$

$$\text{whence } x = 2\sqrt{y} = 16, \text{ or } -24.$$

Again, taking the value, $2x = -\frac{1}{4}\sqrt{y}$;

$$\text{then } y - \frac{1}{4}\sqrt{y} = 96;$$

completing the square,

$$y - \frac{1}{4}\sqrt{y} + \frac{1}{64} = 96 + \frac{1}{64} = \frac{6145}{64};$$

$$\text{extracting the root, } \sqrt{y} - \frac{1}{8} = \frac{\pm\sqrt{(6145)}}{8};$$

$$\text{and } \sqrt{y} = \frac{1 \pm \sqrt{(6145)}}{8};$$

$$\therefore y = \frac{3073 \pm \sqrt{(6145)}}{32};$$

$$\text{and } x = -\frac{1}{8}\sqrt{y} = \frac{-1 \mp \sqrt{(6145)}}{64}.$$

24. Given

$$\left.\begin{array}{l} \sqrt{y} + \sqrt{x} : \sqrt{y} - \sqrt{x} :: \sqrt{x} + 2 : 1 \\ \text{and } \dfrac{\sqrt{y} + 2}{\sqrt{x}} - 1 = \dfrac{3\sqrt{x} + 1 + \dfrac{\sqrt{y}}{\sqrt{x}}}{\sqrt{y}} \end{array}\right\},$$

to find the values of x and y.

From the first equation,

$$\sqrt{y} : \sqrt{x} :: \sqrt{x} + 3 : \sqrt{x} + 1;$$

$$\therefore \sqrt{xy} + \sqrt{y} = x + 3\sqrt{x};$$

and from the second,

$$y + 2\sqrt{y} - \sqrt{xy} = 3x + \sqrt{x} + \sqrt{y};$$

$\therefore$ by transposition, $y + \sqrt{y} - \sqrt{xy} = 3x + \sqrt{x};$

but $\sqrt{y} + \sqrt{xy} = x + 3\sqrt{x};$

$\therefore$ by addition, $y + 2\sqrt{y} = 4x + 4\sqrt{x};$

completing the square, $y + 2\sqrt{y} + 1 = 4x + 4\sqrt{x} + 1,$

and extracting the root, $\sqrt{y} + 1 = \pm(2\sqrt{x} + 1);$

and $\therefore$ if the positive value be taken, $\sqrt{y} = 2\sqrt{x};$

$$\therefore x + 3\sqrt{x} = 2\sqrt{x} + 2x,$$

and $x = \sqrt{x};$

$\therefore \sqrt{x} = 1$, and $x = 1;$

whence $\sqrt{y} = 2\sqrt{x} = 2,$

and $y = 4.$

But if the negative value be taken,

$$\sqrt{y} + 1 = -2\sqrt{x} - 1;$$

by transposition, $\sqrt{y} = -2\sqrt{x} - 2;$

and if this value be substituted in the first equation,

$$-\sqrt{x} - 2 : -3\sqrt{x} - 2 :: \sqrt{x} + 2 : 1,$$

or $\sqrt{x} + 2 : 3\sqrt{x} + 2 :: \sqrt{x} + 2 : 1;$

and since the first term in this proportion is equal to the third, the second will be equal to the fourth;

$$\therefore 3\sqrt{x} + 2 = 1;$$

by transposition, $3\sqrt{x} = -1;$

$\therefore \sqrt{x} = -\frac{1}{3}$, and $x = \frac{1}{9};$

whence $\sqrt{y} = -2 \cdot (\sqrt{x} + 1) = -\frac{4}{3},$

and $y = \frac{16}{9}.$

25. Given $\left.\begin{array}{l} y+\sqrt{\dfrac{y}{x}}=\dfrac{42}{x} \\ \text{and } \dfrac{x^2}{3}+\dfrac{x}{2\sqrt{y}}=\dfrac{54}{y} \end{array}\right\}$, to find the values of x and y.

Completing the square in the first equation,

$$y+\sqrt{\frac{y}{x}}+\frac{1}{4x}=\frac{42}{x}+\frac{1}{4x}=\frac{169}{4x},$$

and extracting the root, $\sqrt{y}+\dfrac{1}{2\sqrt{x}}=\dfrac{\pm 13}{2\sqrt{x}}$;

$$\therefore \sqrt{y}=\frac{6}{\sqrt{x}}, \text{ or } \frac{-7}{\sqrt{x}};$$

$$\therefore y=\frac{36}{x}, \text{ or } \frac{49}{x}.$$

Again, from the second equation,

$$x^2+\frac{3x}{2\sqrt{y}}=\frac{162}{y};$$

completing the square,

$$x^2+\frac{3x}{2\sqrt{y}}+\frac{9}{16y}=\frac{162}{y}+\frac{9}{16y}=\frac{2601}{16y};$$

and extracting the root, $x+\dfrac{3}{4\sqrt{y}}=\pm\dfrac{51}{4\sqrt{y}}$;

$$\therefore x=\frac{12}{\sqrt{y}}, \text{ or } \frac{-27}{2\sqrt{y}}.$$

$$\text{But } \sqrt{y}=\frac{6}{\sqrt{x}}, \text{ or } \frac{-7}{\sqrt{x}};$$

whence, $x=\dfrac{12}{\sqrt{y}}, \text{ or } \dfrac{-27}{2\sqrt{y}}$

$$=2\sqrt{x}, \text{ or } \frac{-12\sqrt{x}}{7}, \text{ or } \frac{-9\sqrt{x}}{4}, \text{ or } \frac{27\sqrt{x}}{14}.$$

$$\therefore \sqrt{x} = 2, \text{ or } \frac{-12}{7}, \text{ or } \frac{-9}{4}, \text{ or } \frac{27}{14};$$

$$\therefore x = 4, \text{ or } \frac{144}{49}, \text{ or } \frac{81}{16}, \text{ or } \frac{729}{196},$$

$$\text{and } y = \frac{36}{x}, \text{ or } \frac{49}{x} = 9, \text{ or } \frac{49}{4}, \text{ or } \frac{784}{81}, \text{ or } \frac{49 \times 196}{729}.$$

26. Given $\left.\begin{aligned} \frac{x^2}{y^2} + \frac{2x+y}{\sqrt{y}} &= 20 - \frac{y^2+x}{y} \\ \text{and } x + 8 &= 4y \end{aligned}\right\}$, to find the values of x and y.

From the first equation, by transposition,

$$\frac{x^2}{y^2} + \frac{2x}{\sqrt{y}} + y + \frac{x}{y} + \sqrt{y} = 20;$$

completing the square,

$$\left(\frac{x}{y} + \sqrt{y}\right)^2 + \left(\frac{x}{y} + \sqrt{y}\right) + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4};$$

$$\text{extracting the root, } \frac{x}{y} + \sqrt{y} + \frac{1}{2} = \pm \frac{9}{2};$$

$$\therefore \frac{x}{y} + \sqrt{y} = 4, \text{ or } -5,$$

$$\text{and } x + y^{\frac{3}{2}} = 4y, \text{ or } -5y.$$

Let the former be taken.

Now from the second equation, $x + 8 = 4y$;

$$\therefore \text{ by subtraction, } y^{\frac{3}{2}} - 8 = 0,$$

$$\text{or } y^{\frac{3}{2}} = 8;$$

$$\therefore y = 4;$$

$$\text{and } x = 4y - 8 = 16 - 8 = 8.$$

$$\text{But if } x + y^{\frac{1}{2}} = -5y,$$
$$\text{and } x + 8 = 4y;$$
$$\therefore \text{ by subtraction, } 8 - y^{\frac{1}{2}} = 9y;$$
$$\therefore \text{ by transposition, } 8 - 8y = y + y^{\frac{1}{2}},$$
$$\text{or } 8 \,.\, (1 - y) = y \,.\, (1 + y^{\frac{1}{2}}).$$
$$\text{Dividing by } (1 + y^{\frac{1}{2}}), \therefore 8\,(1 - y^{\frac{1}{2}}) = y;$$
$$\therefore \text{ by transposition, } y + 8y^{\frac{1}{2}} = 8;$$
$$\text{completing the square, } y + 8y^{\frac{1}{2}} + 16 = 24,$$
$$\text{and extracting the root, } y^{\frac{1}{2}} + 4 = \pm 2\sqrt{6};$$
$$\therefore y^{\frac{1}{2}} = -4 \pm 2\sqrt{6},$$
$$\text{and } y = 40 \mp 16\sqrt{6};$$
$$\therefore x = 4y - 8 = 152 \mp 64\sqrt{6}.$$

27. Given $\left.\begin{array}{l} 8x + 23y = 2x^2 + 2y^2 \\ \text{and } 34y + 6x^2 - 5y^2 = 13xy + 24 \end{array}\right\}$, to find the values of x and y.

From the second equation,

$$6x^2 - 13xy = 5y^2 - 34y + 24,$$
$$\text{and } x^2 - \frac{13}{6} \,.\, xy = \frac{5}{6}y^2 - \frac{34}{6}y + 4;$$

completing the square,

$$x^2 - \frac{13}{6}xy + \frac{169}{144}y^2 = \frac{169}{144}y^2 + \frac{5}{6}y^2 - \frac{34}{6}y + 4$$
$$= \frac{289}{144} \,.\, y^2 - \frac{34}{6} \,.\, y + 4;$$

$$\text{extracting the root, } x - \frac{13}{12}y = \pm\left(\frac{17}{12} \,.\, y - 2\right);$$

$$\text{and first, taking the positive value, } x = \frac{5}{2}y - 2.$$

Let this value of x be substituted in the first equation,

$$\therefore\ 20y - 16 + 23y = \frac{125y^3}{4} - 75y^2 + 60y - 16 + 2y^3;$$

$$\therefore \text{ by transposition, } \frac{133y^3}{4} - 75y^2 + 17y = 0,$$

$$\text{and dividing by } \frac{133y}{4},\ y^2 - \frac{300}{133}y + \frac{68}{133} = 0;$$

$$\text{by transposition, } y^2 - \frac{300}{133}y = -\frac{68}{133};$$

completing the square,

$$y^2 - \frac{300}{133}\cdot y + \left(\frac{150}{133}\right)^2 = \frac{22500}{(133)^2} - \frac{68}{133} = \frac{13456}{(133)^2};$$

$$\text{extracting the root, } y - \frac{150}{133} = \pm\frac{116}{133};$$

$$\therefore\ y = 2, \text{ or } \frac{34}{133};$$

$$\therefore\ x = \frac{5y}{2} - 2 = 3, \text{ or } \frac{-171}{133}.$$

But if the negative value be taken,

$$x = 2 - \frac{1}{3}y.$$

Let this value be substituted in the first equation,

$$\therefore\ 16 - \frac{8}{3}y + 23y = 16 - 8y + \frac{4}{3}y^2 - \frac{2y^3}{27} + 2y^3;$$

$$\therefore \text{ by transposition, } \frac{52y^3}{27} + \frac{4}{3}y^2 = \frac{85}{3}y,$$

$$\text{and dividing by } \frac{52y}{27},\ y^2 + \frac{9}{13}y = \frac{765}{52};$$

completing the square,

$$y^2 + \frac{9}{13}y + \left(\frac{9}{26}\right)^2 = \left(\frac{9}{26}\right)^2 + \frac{765}{52} = \frac{10026}{(26)^2};$$

$$\text{extracting the root, } y + \frac{9}{26} = \frac{\pm\sqrt{(10026)}}{26};$$

$$\therefore\ y = \frac{-9 \pm \sqrt{(10026)}}{26} = \frac{-9 \pm 3\sqrt{(1114)}}{26},$$

$$\text{and } x = 2 - \frac{1}{3}y = \frac{55 \mp \sqrt{(1114)}}{26}.$$

28. Given $\left.\begin{aligned} \frac{x}{y} - 8\sqrt{(x^2 - 9xy^2)} &= 9y - 16xy \\ \text{and } 5x &= 4 + 25y^2 \end{aligned}\right\}$, to find the values of x and y.

From the first equation,

$$x - 8x^{\frac{1}{2}}y\sqrt{(x - 9y^2)} = 9y^2 - 16xy^2;$$

$\therefore$ by transposition,

$$(x - 9y^2) - 8x^{\frac{1}{2}}y\sqrt{(x - 9y^2)} + 16xy^2 = 0;$$

$$\text{extracting the root, } \sqrt{(x - 9y^2)} - 4x^{\frac{1}{2}}y = 0,$$

$$\text{and } \sqrt{(x - 9y^2)} = 4x^{\frac{1}{2}}y,$$

$$\text{or } x - 9y^2 = 16xy^2;$$

$$\text{and } x = (16x + 9)\cdot y^2;$$

$$\therefore\ y^2 = \frac{x}{16x + 9}.$$

But from the second equation, $y^2 = \dfrac{5x - 4}{25}$;

$$\therefore\ \frac{5x - 4}{25} = \frac{x}{16x + 9},$$

$$\text{and } 80x^2 - 19x - 36 = 25x;$$

$$\therefore \text{ by transposition, } 80x^2 - 44x = 36,$$

$$\text{and } x^2 - \frac{22}{40}\cdot x = \frac{18}{40};$$

completing the square,

$$x^2 - \frac{22}{40}\cdot x + \left(\frac{11}{40}\right)^2 = \frac{121}{1600} + \frac{18}{40} = \frac{841}{1600};$$

$$\text{extracting the root, } x - \frac{11}{40} = \pm\frac{29}{40};$$

$$\therefore\ x = 1, \text{ or } -\frac{9}{20};$$

$$\therefore\ y^2 = \frac{5x-4}{25} = \frac{1}{25},\ \text{or} - \frac{1}{4};$$

$$\therefore\ y = \pm\frac{1}{5},\ \text{or} \pm\frac{1}{2}\sqrt{-1}.$$

29. Given $\left.\begin{array}{l} 16x - y^{\frac{1}{2}} = 6y^{\frac{1}{4}}x^{\frac{1}{2}} \\ \text{and } \dfrac{x^4}{y} - \dfrac{12}{x^2} = \dfrac{x}{\sqrt{y}} \end{array}\right\}$, to find the values of x and y.

From the first equation by transposition,

$$16x = y^{\frac{1}{2}} + 6y^{\frac{1}{4}}x^{\frac{1}{2}};$$

completing the square, $25x = y^{\frac{1}{2}} + 6y^{\frac{1}{4}}x^{\frac{1}{2}} + 9x$;

extracting the root, $\pm 5x^{\frac{1}{2}} = y^{\frac{1}{4}} + 3x^{\frac{1}{2}}$;

$$\therefore\ 2x^{\frac{1}{2}},\ \text{or} - 8x^{\frac{1}{2}} = y^{\frac{1}{4}},$$

and therefore $4x$, or $64x = y^{\frac{1}{2}}$.

Now from the second equation, $\dfrac{x^4}{y} - \dfrac{x}{\sqrt{y}} = \dfrac{12}{x^2}$;

completing the square, $\dfrac{x^4}{y} - \dfrac{x}{\sqrt{y}} + \dfrac{1}{4x^2} = \dfrac{12}{x^2} + \dfrac{1}{4x^2} = \dfrac{49}{4x^2}$;

extracting the root, $\dfrac{x^2}{\sqrt{y}} - \dfrac{1}{2x} = \pm\dfrac{7}{2x}$;

$$\therefore\ \frac{x^2}{\sqrt{y}} = \frac{4}{x},\ \text{or} - \frac{3}{x},$$

$$\text{and } \frac{x^3}{\sqrt{y}} = 4,\ \text{or} - 3.$$

Now $\sqrt{y} = 4x$, or $64x$;

which, substituted in the last equation, gives,

$$\frac{x^2}{4},\ \text{or}\ \frac{x^2}{64} = 4,\ \text{or} - 3;$$

$$\therefore\ \frac{x}{2},\ \text{or}\ \frac{x}{8} = \pm 2,\ \text{or} \pm\sqrt{-3};$$

whence $x = \pm 4$, or ± 16, or $\pm 2\sqrt{-3}$, or $\pm 8\sqrt{-3}$,

and $y = 256$, or $\overline{256}|^{2}$, or -192, or $-3 \times (64)^{4}$.

30. Given $\left.\begin{aligned} y^2 - 64 &= 8x^{\frac{1}{2}}y \\ \text{and } y - 4 &= 2y^{\frac{1}{2}}x^{\frac{1}{2}} \end{aligned}\right\}$, to find the values of x and y.

From the first equation, by transposition,

$$y^2 - 8x^{\frac{1}{2}}y = 64;$$

completing the square, $y^2 - 8x^{\frac{1}{2}}y + 16x = 16x + 64$;

extracting the root, $y - 4x^{\frac{1}{2}} = \pm 4\sqrt{(x + 4)}$,

and $y = 4x^{\frac{1}{2}} \pm 4\sqrt{(x + 4)}$.

Also from the second equation, $y - 2y^{\frac{1}{2}}x^{\frac{1}{2}} = 4$;

completing the square, $y - 2y^{\frac{1}{2}}x^{\frac{1}{2}} + x = x + 4$;

extracting the root, $y^{\frac{1}{2}} - x^{\frac{1}{2}} = \pm\sqrt{(x + 4)}$;

$\therefore\ 4y^{\frac{1}{2}} = 4x^{\frac{1}{2}} \pm 4\sqrt{(x + 4)} = y$, from the last equation;

$\therefore\ 4 = y^{\frac{1}{2}}$, and $16 = y$;

And from the second equation, $x^{\frac{1}{2}} = \dfrac{y - 4}{2y^{\frac{1}{2}}} = \dfrac{12}{8} = \dfrac{3}{2}$;

$$\therefore\ x = \frac{9}{4}.$$

31. Given $\left.\begin{aligned} \sqrt{(5\sqrt{x} + 5\sqrt{y})} + \sqrt{y} &= 10 - \sqrt{x} \\ \text{and } \sqrt{x^5} + \sqrt{y^5} &= 275 \end{aligned}\right\}$, to find the values of x and y.

From the first equation,

$$\sqrt{x} + \sqrt{y} + \sqrt{5}\,.\sqrt{(\sqrt{x} + \sqrt{y})} = 10;$$

completing the square,

$$\left(\sqrt{x} + \sqrt{y}\right) + \sqrt{5}\,.\sqrt{(\sqrt{x} + \sqrt{y})} + \frac{5}{4} = 10 + \frac{5}{4} = \frac{45}{4};$$

extracting the root, $\sqrt{(\sqrt{x} + \sqrt{y})} + \dfrac{\sqrt{5}}{2} = \pm\dfrac{3\sqrt{5}}{2}$,

and $\sqrt{(\sqrt{x} + \sqrt{y})} = \sqrt{5}$, or $-2\sqrt{5}$;

$\therefore \sqrt{x} + \sqrt{y} = 5$, or 20, supposing the former,

$\therefore$ by involution,

$$x^{\frac{5}{2}} + 5x^2y^{\frac{1}{2}} + 10x^{\frac{3}{2}}y + 10xy^{\frac{3}{2}} + 5x^{\frac{1}{2}}y^2 + y^{\frac{5}{2}} = 3125;$$

but $x^{\frac{5}{2}} + y^{\frac{5}{2}} = 275$;

$\therefore$ by subtraction,

$$5x^2y^{\frac{1}{2}} + 10x^{\frac{3}{2}}y + 10xy^{\frac{3}{2}} + 5x^{\frac{1}{2}}y^2 = 2850;$$

or $5x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{3}{2}} + 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y + y^{\frac{3}{2}}) = 2850$;

and $x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{3}{2}} + 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y + y^{\frac{3}{2}}) = 570$;

but $x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{3}{2}} + 3xy^{\frac{1}{2}} + 3x^{\frac{1}{2}}y + y^{\frac{3}{2}}) = 125x^{\frac{1}{2}}y^{\frac{1}{2}}$;

$\therefore$ by subtraction, $x^{\frac{1}{2}}y^{\frac{1}{2}} \times (xy^{\frac{1}{2}} + x^{\frac{1}{2}}y) = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570$;

or $x^{\frac{1}{2}}y^{\frac{1}{2}} \times x^{\frac{1}{2}}y^{\frac{1}{2}} \times (x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570$;

or $5xy = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570$;

$\therefore xy - 25x^{\frac{1}{2}}y^{\frac{1}{2}} = -114$;

completing the square,

$$xy - 25x^{\frac{1}{2}}y^{\frac{1}{2}} + \left(\frac{25}{2}\right)^2 = \frac{625}{4} - 114 = \frac{169}{4};$$

extracting the root, $x^{\frac{1}{2}}y^{\frac{1}{2}} - \frac{25}{2} = \pm\frac{13}{2}$;

$\therefore x^{\frac{1}{2}}y^{\frac{1}{2}} = 19$, or 6;

but $x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 25$,

and $4x^{\frac{1}{2}}y^{\frac{1}{2}} = 24$, or 76;

$\therefore$ by subtraction, $x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 1$, or -51;

extracting the root, $x^{\frac{1}{2}} - y^{\frac{1}{2}} = \pm 1$, or $\pm\sqrt{(-51)}$;

but $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5$;

$\therefore$ by addition, $2x^{\frac{1}{2}} = 6$, or 4, or $5 \pm \sqrt{(-51)}$;

$$\therefore\ x^{\frac{1}{2}} = 3, \text{ or } 2, \text{ or } \frac{5 \pm \sqrt{(-51)}}{2},$$

$$\text{and } x = 9, \text{ or } 4, \text{ or } \frac{-13 \pm \sqrt{(-51)}}{2};$$

by subtraction, $2y^{\frac{1}{2}} = 4, \text{ or } 6, \text{ or } 5 \mp \sqrt{(-51)}$;

$$\therefore\ y^{\frac{1}{2}} = 2, \text{ or } 3, \text{ or } \frac{5 \mp \sqrt{(-51)}}{2};$$

$$\therefore\ y = 4, \text{ or } 9, \text{ or } \frac{-13 \mp \sqrt{(-51)}}{2}.*$$

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* The other case, where $\sqrt{x} + \sqrt{y} = 20$, is solved in the same manner.

SECTION VI.

On the Solution of Problems which involve Simple Equations.

(31.) THE solution of a problem, or method of discovering by analysis quantities which will answer its several conditions, is performed by assuming algebraic symbols to represent the quantities sought, and by deducing equations from the application of these, in the same manner as if they were known quantities, to the conditions of the problem. The independent equations derived from this process, if the conditions be properly limited, will equal in number the unknown quantities assumed; and from the solution of these several equations by the rules already given (23. 28, 29, 30), the values of the algebraic symbols will be determined. Whether these values are correct, may be determined synthetically, by applying them instead of their respective symbols to the several conditions of the problem.

If the conditions of the problem are not properly limited, that is, are not sufficient in number, or not sufficiently independent of each other, the resulting equations will either exceed in number the unknown quantities, and will therefore some of them be identical or inconsistent, or will be fewer in number than the unknown quantities, and consequently will admit of an indefinite number of answers; see Section XI.

In many cases, instead of assuming a symbol to represent each of the required quantities, it is convenient to assume one only, and from the conditions of the problem to deduce expressions for the others in terms of that one and known quantities. And as the number of conditions ought to be one more than the number of quantities thus expressed, there will remain one to be stated in an equation; from which the value of the unknown quantity may be determined (22. 28, 29): and this being substituted in the other expressions, their value also may be discovered.

Examples of the Solution of Problems producing Simple Equations involving only one unknown quantity.

1. What number is that, to the double of which if 18 be added the sum will be 82?

Let $x =$ the number required.
Then by the problem, $2x + 18 = 82$;
$\therefore$ by transposition, $2x = 64$,
and $x = 32$.

2. What number is that, to the double of which if 44 be added the sum is equal to four times the required number?

Let $x =$ the number.
Then $2x + 44 = 4x$, by supposition;
$\therefore$ by transposition, $44 = 2x$,
and $22 = x$.

3. What number is that, the double of which exceeds its half by 6?

Let $x =$ the number.

Then by the problem, $2x - \frac{x}{2} = 6$,

$$\therefore 4x - x = 12,$$
$$\text{or } 3x = 12,$$
$$\therefore x = 4.$$

4. From two towns which are 187 miles distant, two travellers set out at the same time with an intention of meeting. One of them goes 8 miles, and the other 9 miles a day. In how many days will they meet?

Let $x =$ the number of days required;
then $8x =$ the number of miles one travelled,
and $9x =$ the number the other travelled;

and since they meet, they must together have travelled the whole distance,

$$\text{consequently } 8x + 9x = 187,$$
$$\text{or } 17x = 187,$$
$$\therefore x = 11.$$

5. A Gentleman meeting 4 poor persons distributed five shillings amongst them: to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. What did he give to each?

Let $x =$ the pence he gave to the first,
$\therefore 2x =$ the pence given to the second,
and $3x =$ - - - - - - - - - - to the third,
$4x =$ - - - - - - - - - - to the fourth.

$$\therefore x + 2x + 3x + 4x = 60,$$
$$\text{or } 10x = 60,$$
$$\therefore x = 6,$$

and $\therefore$ he gave 6, 12, 18, 24 pence respectively to them.

6. A Bookseller sold 10 books at a certain price; and afterwards 15 more at the same rate. Now at the latter time he received 35 shillings more than at the former. What did he receive for each book?

Let $x =$ the price of a book, in shillings,
Then $10x =$ price of the first set,
and $15x =$ price of the second set.
But by the problem $15x = 10x + 35$;
$\therefore$ by transposition, $5x = 35$,
and $x = 7$.

7. A Gentleman dying bequeathed a legacy of £140 to three servants. *A* was to have twice as much as *B*; and *B* three times as much as *C*. What were their respective shares?

Let $x = C$'s share,
$\therefore 3x = B$'s share,
and $6x = A$'s share;
whence $(6x + 3x + x =)\ 10x = 140$,
$\therefore x = 14$,
A $\therefore$ received £84; *B*, £42; and *C* £14.

8. Four Merchants entered into a speculation, for which they subscribed £4755; of which *B* paid three times as much as *A*; *C* paid as much as *A* and *B*; and *D* paid as much as *C* and *B*. What did each pay?

Let $x =$ number of pounds A paid;
$\therefore\ 3x =$ number B paid,
$4x =$ number C paid,
and $7x =$ number D paid;
$\therefore\ (x + 3x + 4x + 7x =)\ 15x = 4755$,
and $\therefore\ x = 317$.

$\therefore$ they contributed 317, 951, 1268, and 2219 pounds respectively.

9. A Draper bought three pieces of cloth, which together measured 159 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What was the length of each?

Let $x =$ the number of yards in the first piece,
$\therefore\ x + 15 =$ the number in the second,
and $x + 39 =$ the number in the third.
$\therefore\ x + x + 15 + x + 39 = 159$,
and by transposition, $3x = 105$,
$\therefore\ x = 35$,

$\therefore$ the lengths are 35, 50, and 74 yards respectively.

10. A cask which held 146 gallons, was filled with a mixture of brandy, wine, and water. In it there were 15 gallons of wine more than there were of brandy, and within 4 gallons as much water as both wine and brandy. What quantity was there of each?

Let $x =$ the number of gallons of brandy,
$\therefore\ x + 15 =$ number of gallons of wine,
and $2x + 11 =$ number of gallons of water.
$\therefore\ x + x + 15 + 2x + 11 = 146$,
$\therefore$ by transposition, $4x = 120$,
and $x = 30$.

$\therefore$ there were 30, 45, and 71 gallons respectively of brandy, wine, and water.

11. A person employed 4 workmen; to the first of whom he gave 2 shillings more than to the second; to the second

3 shillings more than to the third; and to the third 4 shillings more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

Let $x =$ the sum received by the fourth,

$\therefore\ x + 4 =$ - - - - - - - - - - - - third,

$x + 7 =$ - - - - - - - - - - - - second,

and $x + 9 =$ - - - - - - - - - - - - first.

$$\therefore\ x + x + 4 + x + 7 + x + 9 = 32,$$

and by transposition, $4x = 12$,

$$\therefore\ x = 3.$$

$\therefore$ they received 12, 10, 7, and 3 shillings respectively.

12. A Father taking his 4 sons to school, divided a certain sum amongst them. Now the third had 9 shillings more than the youngest; the second 12 shillings more than the third; and the eldest 18 shillings more than the second; and the whole sum was 6 shillings more than 7 times the sum which the youngest received. How much had each?

Suppose the youngest received x shillings,

then the third received $x + 9$ - - - -

the second - - - - $x + 21$ - - - -

and the eldest - - - - $x + 39$ - - - -

$$\therefore\ x + x + 9 + x + 21 + x + 39 = 7x + 6,$$

$\therefore$ by transposition, $63 = 3x$,

and $\therefore\ 21 = x$,

consequently they received 21, 30, 42, and 60 shillings respectively.

13. A sum of money was to be divided amongst six poor persons; the second received 10*d.* the third 14*d.* the fourth 25*d.* the fifth 28*d.* and the sixth 33*d.* less than the first. Now the sum distributed was 10*d.* more than the treble of what the first received. What did each receive?

$$\begin{aligned} \text{Let } x &= \text{what the first received,} \\ \therefore\ x - 10 &= \text{- - - - second - - -} \\ x - 14 &= \text{- - - - third - - -} \\ x - 25 &= \text{- - - - fourth - - -} \\ x - 28 &= \text{- - - - fifth - - -} \\ x - 33 &= \text{- - - - sixth - - -} \end{aligned}$$

The sum of which $= 6x - 110 = 3x + 10$ by supposition;

$\therefore$ by transposition, $3x = 120$,

and $x = 40$.

$\therefore$ they received 40, 30, 26, 15, 12, 7 pence respectively.

14. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3; be less than the third by 10; greater than the fourth by 9; and less than the fifth by 16.

$$\begin{aligned} \text{Let } x &\doteq \text{the first part,} \\ \therefore\ x - 3 &= \text{second,} \\ x + 10 &= \text{third,} \\ x - 9 &= \text{fourth,} \\ x + 16 &= \text{fifth.} \end{aligned}$$

$\therefore\ x + x - 3 + x + 10 + x - 9 + x + 16 = 99$,

or $5x + 14 = 99$,

$\therefore$ by transposition, $5x = 85$,

and $x = 17$.

$\therefore$ the parts are 17, 14, 27, 8, and 33.

15. What two numbers are those whose sum is 59, and difference 17?

Let $x =$ the less,

$\therefore\ x + 17 =$ the greater,

and $\therefore\ x + x + 17 = 59$,

by transposition, $2x = 42$.

and $x = 21$ the less,

$\therefore$ the greater $= 38$.

16. What number is that, the treble of which increased by 12, shall as much exceed 54 as that treble is below 144?

Let $x =$ the number.

$\therefore\ 3x + 12 - 54 = 144 - 3x$ by supposition;

$\therefore$ by transposition, $6x = 186$,

and $x = 31$.

17. Two persons began to play with equal sums of money: the first lost 14 shillings, the other won 24 shillings; and then the second had twice as many shillings as the first. What sum had each at first?

Let $x =$ the sum;

$$\left.\begin{array}{r}\text{Then } x - 14 \\ \text{and } x + 24\end{array}\right\} = \text{the sums each had after playing;}$$

$\therefore$ by the problem $2x - 28 = x + 24$;

$\therefore\ x = 52$.

18. At a certain election 943 men voted, and the candidate chosen had a majority of 65. How many voted for each?

Let $x =$ the number of votes the unsuccessful candidate had;

$\therefore\ x + 65 =$ the number the successful one had.

$\therefore\ x + x + 65 = 943$;

by transposition, $2x = 878$,

and $x = 439$.

$\therefore$ the numbers were 439 and 504.

19. Two Robbers after plundering a house found that they had 35 guineas between them; and that if one of them had had 4 guineas more, he should have had twice as many as the other. How many had each?

Let $x =$ the number one had,

$\therefore\ 35 - x =$ the number the other had,

and $35 - x + 4 = 2x$.

by transposition, $39 = 3x$,

and $13 = x$.

$\therefore$ they had 13 and 22 guineas respectively.

20. A Mercer having cut 19 yards from each of three equal pieces of silk, and 17 from another of the same length, found that the remnants taken together were 142 yards. What was the length of each piece?

Let x = the length,

$\therefore\ x - 19$ = the length of each of the 3 equal remnants,

and $x - 17$ = the length of the other.

$$\text{then } 3 \,.\, (x - 19) + x - 17 = 142,$$
$$\text{or } 3x - 57 + x - 17 = 142.$$
$$\text{by transposition, } 4x = 216;$$
$$\therefore\ x = 54.$$

21. A Farmer has two flocks of sheep, each containing the same number. From one of these he sells 39, and from the other 93; and finds just twice as many remaining in one as in the other. How many did each flock originally contain?

Let x = the number required.

Then $x - 39$, and $x - 93$, are the numbers remaining;

$$\therefore\ x - 39 = 2x - 186;$$
$$\text{and by transposition, } 147 = x.$$

22. Bought 12 yards of cloth for £10. 14*s*. For part of it I gave 19 shillings a yard, and for the rest 17 shillings a yard. How many yards of each were bought?

Let x = the number of yards at 19*s*. per yard;

$\therefore\ 12 - x$ = the number at 17*s*.

and $19x$ = the price of the former,

and $17 \,.\, (12 - x)$ = the price of the latter.

$$\therefore\ 19x + 204 - 17x = 214.$$
$$\text{and by transposition, } 2x = 10,$$
$$\text{and } x = 5.$$

$\therefore$ there were 5 yards at 19 shillings, and 7 at 17 shillings.

23. Divide the number 197 into two such parts, that four times the greater may exceed five times the less by 50.

Let $x =$ the less,
and $\therefore\ 197 - x =$ the greater.
Then $788 - 4x = 5x + 50$;
and by transposition, $738 = 9x$,
and $82 = x$;
$\therefore$ the greater $= 115$.

24. A Courier, who travels 60 miles a day, had been dispatched 5 days, when a second was sent to overtake him; in order to which, he must go 75 miles a day: In what time will he overtake the former?

Let $x =$ the number of days the second courier travels;
then $x + 5 =$ the number the first travels;
$\therefore\ 75x =$ the number of miles the second travels,
and $60.(x+5) =$ the number the first travels.

But by the supposition they both travelled the same number of miles;

$$\therefore\ 75x = 60x + 300,$$
by transposition, $15x = 300$,
and $x = 20$.

25. After A had lost 10 guineas to B, he wanted only 8 guineas in order to have as much money as B; and together they had 60 guineas. What money had each at first?

Let $x =$ the number of guineas A had;
$\therefore\ 60 - x =$ the number - - - - B had.
Then after playing A had $x - 10$, and B had $70 - x$;

$$\therefore\ x - 10 + 8 = 70 - x,$$
by transposition, $2x = 72$,
and $x = 36$.
$\therefore$ they had 36 and 24 guineas respectively.

26. A and B began trade with equal stocks. In the first year A tripled his stock, and had £27 to spare; B doubled his stock, and had £153 to spare. Now the amount of both their gains was five times the stock of either. What was that stock?

Let $x =$ the stock;
then $3x + 27 = A$'s stock at the end of the year,
$\therefore\ 2x + 27 =$ his gain,
and $2x + 153 = B$'s stock at the end of the year;
$\therefore\ x + 153 = B$'s gain;
$\therefore\ 5x = 2x + 27 + x + 153.$
by transposition, $2x = 180$,
and $x = 90$.

27. Two workmen A and B were employed together for 50 days, at 5 shillings per day each. A spent sixpence a day less than B did, and at the end of the fifty days he found he had saved twice as much as B, and the expense of two days over. What did each spend per day?

Let $x =$ what A spent per day (in pence);
$\therefore\ 60 - x =$ what he saved per day,
and $54 - x =$ what B saved per day.
and $\therefore\ 3000 - 50x = 5400 - 100x + 2x.$
by transposition, $48x = 2400$,
and $x = 50$;
$\therefore\ A$ spent 50 pence, and B 56 pence a day.

28. A and B began to trade with equal sums of money. In the first year A gained 40 pounds and B lost 40; but in the second A lost one-third of what he then had, and B gained a sum less by 40 pounds than twice the sum that A had lost; when it appeared that B had twice as much money as A. What money did each begin with?

Let $x =$ the number of pounds each had at first.
then $x + 40 =$ the sum A had after the first year,
and $x - 40 =$ the sum B had,
also $\frac{2}{3}\,.\,(x + 40) =$ the sum A had after the 2[d] year,
and $x - 40 + \frac{2}{3}(x + 40) - 40 =$ the sum B had;
$\therefore\ \frac{4}{3}\,.\,(x + 40) = x - 40 + \frac{2}{3}\,.\,(x + 40) - 40,$
and $\frac{2}{3}\,.\,(x + 40) = x - 80$;
$\therefore\ 2x + 80 = 3x - 240,$
and by transposition, $320 = x$.

29. Divide the number 68 into two such parts, that the difference between the greater and 84 may equal three times the difference between the less and 40.

$$\text{Let } x = \text{the less,}$$
$$\text{then } 68 - x = \text{the greater;}$$
$$\therefore 84 - (68 - x) = 3 \cdot (40 - x),$$
$$\text{or } 16 + x = 120 - 3x.$$
$$\text{by transposition, } 4x = 104,$$
$$\text{and } x = 26;$$
$$\text{and } \therefore \text{ the greater} = 42.$$

30. *A* and *B* being at play severally cut packs of cards so as to take off more than they left. Now it happened that *A* cut off twice as many as *B* left, and *B* cut off seven times as many as *A* left. How were the cards cut by each?

$$\text{Suppose } A \text{ cut off } 2x \text{ cards,}$$
$$\text{then } 52 - 2x = \text{the number he left,}$$
$$\text{and } x = \text{the number } B \text{ left;}$$
$$\therefore 52 - x = \text{the number he cut off;}$$
$$\text{whence } 52 - x = 364 - 14x;$$
$$\text{by transposition, } 13x = 312,$$
$$\text{and } x = 24;$$
$$\therefore A \text{ cut off } 48, \text{ and } B \text{ cut off } 28 \text{ cards.}$$

31. What number is that whose one-third part exceeds its one-fourth part by 16?

$$\text{Let* } 12x = \text{the number;}$$
$$\therefore 4x - 3x = 16,$$
$$\text{or } x = 16;$$
$$\text{and } \therefore \text{ the number} = 12 \times 16 = 192.$$

32. Upon measuring the corn produced by a field, being 48 quarters; it appeared that it yielded only one-third part more than was sown. How much was that?

* 12 being the least common multiple of 3 and 4.

Let $3x =$ the number of quarters sown,
then $3x + x = 48$,
or $4x = 48$,
and $x = 12$;
$\therefore$ the quantity sown was 36 quarters.

33. A Farmer sold 96 loads of hay to two persons. To the first one-half, and to the second one-fourth of what his stack contained. How many loads did that stack contain?

Let $4x =$ the number of loads,
then $2x =$ the number the first bought,
and $x =$ the number the second had.
$\therefore (2x + x =) \, 3x = 96$,
and $x = 32$.
whence, the stack contained 128 loads.

34. A Gentleman bequeathed £210 to two servants; to one he left half as much as to the other. What did he leave to each?

Let $2x =$ the sum one received;
$\therefore x =$ the sum left to the other.
$\therefore (2x + x =) \, 3x = 210$,
and $x = 70$;
$\therefore$ they had 140 and 70 pounds respectively.

35. A prize of £2329 was divided between two persons A and B, whose shares therein were in proportion of 5 to 12. What was the share of each?

Let $5x = A$'s share,
then $12x = B$'s share;
$\therefore (5x + 12x =) \, 17x = 2329$,
and $x = 137$;
$\therefore$ their shares were 685 and 1644 pounds respectively.

36. A sum of money is to be shared between two persons A and B, so that as often as A receives 9 pounds, B takes 4. Now it happens that A receives 15 pounds more than B. What are their respective shares?

Since for every £9 that A receives, B receives £4,

Let $9x =$ the whole sum A receives;

$\therefore\ 4x =$ the whole sum B receives;

$$\therefore\ 9x = 4x + 15;$$

and by transposition, $5x = 15$;

$$\therefore\ x = 3;$$

$\therefore$ A receives £27, and B £12.

37. A Gentleman gave to 3 persons 98 pounds. The second received five-eighths of the sum given to the first, and the third one-fifth of what the second had. What did each receive?

Let $8x =$ the number of pounds the first received;

$\therefore\ 5x =$ - - - - - - - - - - - - - - second - - -

and $x =$ - - - - - - - - - - - - - - third - - -

$$\therefore\ (8x + 5x + x =)\ 14x = 98,$$

and $x = 7$;

$\therefore$ they received 56, 35, and 7 pounds, respectively.

38. A person bought two casks of beer, one of which held exactly three times as much as the other. From each of these he drew four gallons, and then found that there were four times as many gallons remaining in the larger, as in the other. How many were there in each at first?

Let $3x =$ the number of gallons in the larger;

and $\therefore$ $x =$ the number in the smaller;

$$\therefore\ 4\,.\,(x - 4) = 3x - 4;$$

by transposition, $x = 12$;

$\therefore$ they held 36 and 12 gallons, respectively.

39. A man at a party at cards betted three shillings to two upon every deal. After twenty deals he won five shillings. How many deals did he win?

Let $x =$ the number of deals he won;

$\therefore\ 20 - x =$ the number he lost;

also $2x =$ the money won,

and $3\,.\,(20 - x) =$ the money lost;

$$\text{whence } 2x - 3\,.\,(20 - x) = 5;$$
$$\therefore \text{ by transposition, } 5x = 65,$$
$$\text{and } x = 13.$$

40. What two numbers are as 2 to 3; to each of which if 4 be added, the sums will be as 5 to 7?

Let $2x$ and $3x$ be the numbers;

$$\therefore 2x + 4 : 3x + 4 :: 5 : 7,$$
$$\text{and (21) } 14x + 28 = 15x + 20;$$
$$\text{by transposition, } 8 = x;$$
$$\text{and } \therefore \text{ the numbers are 16 and 24.}$$

41. A sum of money was divided between two persons A and B, so that the share of A was to that of B as 5 to 3, and exceeded five-ninths of the whole sum by 50 pounds. What was the share of each person?

$$\text{Let } 5x = A\text{'s share};$$
$$\therefore 3x = B\text{'s share},$$
$$\text{and } 8x = \text{the whole sum};$$
$$\therefore 5x = \tfrac{5}{9}\,.\,8x + 50,$$
$$\text{or } x = \tfrac{8}{9}\,.\,x + 10,$$
$$\text{and } 9x = 8x + 90;$$
$$\therefore \text{ by transposition, } x = 90,$$

and the sums were 450 and 270 pounds.

42. Being sent to market to buy a certain quantity of meat, I found that if I bought beef, which was then 4 pence a pound, I should lay out all the money I was entrusted with; but if I bought mutton which was then threepence halfpenny a pound, I should have two shillings left. How much meat was sent for?

$$\text{Let } 2x = \text{the number of pounds};$$
$$\therefore 8x = \text{the price of } 2x \text{ lbs. of beef},$$
$$\text{and } 7x = \text{the price of } 2x \text{ lbs. of mutton},$$
$$\text{and } 8x = 7x + 24;$$
$$\therefore x = 24;$$

whence 48 lbs. were sent for.

43. A Fish was caught, whose tail weighed 9 lbs.; his head weighed as much as his tail, and half his body; and his body weighed as much as his head and tail. What did the fish weigh?

Let $2x =$ the number of lbs. the body weighed;
then $9 + x =$ the weight of the head;

$$\therefore 9 + 9 + x = 2x;$$

by transposition, $18 = x$;

$\therefore$ the fish weighed $36 + 27 + 9 = 72$ lbs.

44. The joint stock of two partners whose particular shares differed by 40 pounds was to the share of the lesser as 14 to 5. Required the shares.

Suppose $14x =$ the joint stock;
$\therefore 5x =$ the less,
and $9x =$ the greater;

$$\therefore 9x = 5x + 40;$$

by transposition, $4x = 40$,
and $x = 10$;

$\therefore$ the shares are 90 and 50 pounds, respectively.

45. A Bankrupt owed to two creditors 140 pounds; the difference of the debts was to the greater as 4 to 9. What were the debts?

Let $4x =$ the difference of the debts;
$\therefore 9x =$ the greater,
and $5x =$ the less;

$$\therefore (9x + 5x =)\ 14x = 140,$$

and $x = 10$;

$\therefore$ the debts are 90 and 50 pounds.

46. A Gentleman employed two labourers at different times, one for 3 shillings, and the other for 5 shillings a day. Now the number of days added together was 40; and they each received the same sum. How many days was each employed?

Let $x =$ the number of days the second was employed;
$\therefore\ 40 - x =$ the number the first was employed;
and $5x =$ the sum received by the second,
and $3 \,.\, (40 - x) =$ the sum received by the first;

$$\therefore\ 5x = 3 \,.\, (40 - x);$$

by transposition, $8x = 120$,
and $x = 15$;

$\therefore$ the second was employed 15, and the first 25 days.

47. Some persons agreed to give sixpence each to a waterman for carrying them from London to Gravesend; but with this condition, that for every other person taken in by the way, three pence should be abated in their joint fare. Now the waterman took in three more than a fourth part of the number of the first passengers, in consideration of which he took of them but five pence each. How many persons were there at first?

Let $4x =$ the number of passengers at first;
then $x + 3 =$ the number taken in,
and $3x + 9 =$ the sum deducted from their joint fare;

$$\therefore\ 24x - (3x + 9) = 20x;$$

by transposition, $x = 9$;
consequently there were 36 passengers.

48. In a mixture of wine and cyder, half of the whole + 25 gallons was wine, and one-third of the whole − 5 gallons was cyder. How many gallons were there of each?

Let $6x =$ the number of gallons in all;
$\therefore\ 3x + 25 =$ the number of gallons of wine,
and $2x - 5 =$ the number of gallons of cyder;

$$\therefore\ 6x = 3x + 25 + 2x - 5;$$

by transposition, $x = 20$;
consequently there were 85 gallons of wine, and 35 of cyder.

49. *A* and *B* engaged in trade, *A* with £240, and *B* with £96. *A* lost twice as much as *B*; and upon settling their accounts it appeared that *A* had three times as much remaining as *B*. How much did each lose?

$$\begin{aligned}
\text{Let } x &= \text{what } B \text{ lost;}\\
\therefore\ 96 - x &= \text{what he had remaining;}\\
\text{then } 2x &= \text{what } A \text{ lost,}\\
\text{and } 240 - 2x &= \text{what he had remaining;}
\end{aligned}$$

$$\therefore\ 240 - 2x = 3\,.\,(96 - x)$$

by transposition, $x = 48$;

$\therefore$ A lost £96, and B lost £48.

50. Four places are situated in the order of the four letters A, B, C, D. The distance from A to D is 34 miles, the distance from A to B: distance from C to D :: 2 : 3, and one-fourth of the distance from A to B added to half the distance from C to D is three times the distance from B to C. What are the respective distances?

Let $2x$ = the distance from A to B;

$\therefore$ $3x$ = the distance from C to D.

$$\text{and } \left(\frac{x}{2} + \frac{3x}{2} = \right) 2x = 3\,.\,BC;$$

$$\therefore\ BC = \frac{2}{3}x,$$

$$\text{and } \left(2x + 3x + \frac{2}{3}.x = \right)\frac{17x}{3} = 34;$$

$$\therefore\ \frac{x}{3} = 2, \text{ and } x = 6;$$

whence $AB = 12$, $BC = 4$, and $CD = 18$.

51. A Field of wheat and oats which contained 20 acres was put out to a labourer to reap for six guineas, the wheat at 7 shillings an acre, and the oats at 5 shillings. Now the labourer falling ill, reaped only the wheat. How much money ought he to receive according to the bargain?

$$\begin{aligned}
\text{Let } x &= \text{the number of acres of wheat;}\\
\text{then } 20 - x &= \text{the number of acres of oats;}\\
\text{and } 7x &= \text{the price of reaping the wheat (in shillings),}\\
\text{and } 100 - 5x &= \text{the price of reaping the oats;}
\end{aligned}$$

$$\therefore\ 7x + 100 - 5x = 126;$$
$$\text{by transposition, } 2x = 26,$$
$$\text{and } x = 13;$$
$\therefore$ he ought to receive £4. 11*s*.

52. A General having lost a battle, found that he had only half his army + 3600 men left, fit for action; one-eighth of his men + 600 being wounded, and the rest, which were one-fifth of the whole army, either slain, taken prisoners, or missing. Of how many men did his army consist?

Since 40 is the least common multiple of 2, 8, and 5,

let $40x$ = the number required;
$\therefore\ 20x + 3600$ = the number fit for service;
$5x + 600$ = the number wounded,
and $8x$ = the number missing;

$$\therefore\ 40x = 20x + 3600 + 5x + 600 + 8x;$$
$$\text{by transposition, } 7x = 4200,$$
$$\text{and } x = 600.$$

$\therefore$ his army consisted of 24000.

53. Three men, *A*, *B*, and *C*, entered into partnership; *A* paid in as much as *B*, and one-third of *C*; *B* paid in as much as *C*, and one-third of *A*; and *C* paid in £10, and one-third of *A*. What did each man contribute to the stock?

Let $3x$ = the sum *A* contributed;
$\therefore\ 10 + x$ = - - - - *C* - - - - - -
and $10 + 2x$ = - - - - *B* - - - - - -

$$\therefore\ 3x = 10 + 2x + \frac{10 + x}{3};$$
$$\text{by transposition, } \frac{2x}{3} = 10 + \frac{10}{3},$$
$$\text{and } 2x = 40;$$
$$\therefore\ x = 20.$$

and the sums contributed were £60, £50, and £30, by *A*, *B*, *C*, respectively.

54. It is required to divide the number 91 into two such parts that the greater being divided by their difference, the quotient may be 7.

$$\text{Let } x = \text{the greater;}$$
$$\therefore 91 - x = \text{the less,}$$
$$\text{and } \frac{x}{2x - 91} = 7;$$
$$\therefore x = 14x - 637;$$
$$\text{by transposition, } 637 = 13x;$$
$$\text{and } \therefore 49 = x;$$
$$\therefore \text{the parts are } 49 \text{ and } 42.$$

55. From each of 16 coins an artist filed the worth of half a crown, and then offered them in payment for their original value: but being detected, the pieces were found to be really worth no more than 8 guineas. What was their original value?

$$\text{Let } x = \text{the number of sixpences each was worth;}$$
$$\therefore x - 5 = \text{the number each was worth after filing;}$$
$$\therefore 16 \,.\, (x - 5) = 336.$$
$$\text{by transposition, } 16x = 416,$$
$$\text{and } \quad x = 26 = 13 \text{ shillings.}$$

56. *A* and *B* made a joint stock of £833, which, after a successful speculation, produced a clear gain of £153. Of this *B* had £45 more than *A*. What did each person contribute to the stock?

$$\text{Let } x = \text{the sum brought in by } B;$$
$$\text{then } 833 : x :: 153 : B\text{'s gain} = \frac{9x}{49};$$
$$\therefore A\text{'s gain} = \frac{9x}{49} - 45,$$
$$\text{and } \frac{9x}{49} + \frac{9x}{49} - 45 = 153;$$
$$\text{by transposition, } \frac{18x}{49} = 198;$$

$$\therefore x = \frac{49 \times 198}{18} = 49 \times 11 = 539,$$

whence, B brought in £539, and A £294.

57. Sold a quantity of tobacco for 19 shillings, part at 1 shilling a pound, and the rest at 15 pence. Now the first part was to the latter :: $\frac{3}{4}$: $\frac{2}{3}$, How much was sold of each?

$$\text{Since } \tfrac{3}{4} : \tfrac{2}{3} :: 9 : 8,$$

Let $9x$ = the number of lbs. of the former;
$\therefore 8x$ = the number of lbs. of the latter;
$\therefore 9x$ = the number of shillings the first sold for,
and $8x \times \frac{5}{4} = 10x$ = the number of shillings the second sold for.

$$\therefore (10x + 9x =)\ 19x = 19,$$
$$\text{and } x = 1;$$

$\therefore$ there were 9 lbs. at 1 shilling, and 8 lbs. at 15 pence.

58. A Gentleman gave in charity £46; a part thereof in equal portions to 5 poor men, and the rest in equal portions to 7 poor women. Now a man and a woman had between them £8. What was given to the men, and what to the women?

Let $5x$ = the number of pounds the men received;
$\therefore 46 - 5x$ = the number the women received;
$\therefore x$ = the sum one man received,
and $8 - x$ = the sum one woman received;

$$\therefore 56 - 7x = 46 - 5x;$$
$$\text{by transposition, } 2x = 10,$$
$$\text{and } x = 5;$$

$\therefore$ the men received £25, and the women £21.

59. Suppose that for every 10 sheep a farmer kept, he should plough an acre of land, and be allowed one acre of pasture for every 4 sheep. How many sheep may that person keep who farms 700 acres?

Let $x =$ the number of sheep required;

then $10 : x :: 1 :$ the number of acres ploughed $= \frac{x}{10}$,

and $4 : x :: 1 :$ the number of acres of pasture $= \frac{x}{4}$;

$$\therefore \frac{x}{10} + \frac{x}{4} = 700,$$

$$\text{and } (2x + 5x =)\ 7x = 20 \times 700;$$

$$\therefore x = 20 \times 100 = 2000.$$

60. A person being asked the hour, answered that it was between five and six; and the hour and minute-hands were together. What was the time?

Let $x =$ the time past 5;

then since the minute-hand goes 12 times round, whilst the hour-hand goes once, we have this proportion,

$$12 : 1 :: 5 + x : x,$$

$$\text{and } (Alg.\ 179,\ 4.)\ 11 : 1 :: 5 : x;$$

$$\therefore 11x = 5,$$

$$\text{and } x = \tfrac{5}{11} = 27' \cdot 16\tfrac{4}{11}''.$$

61. Divide the number 49 into two such parts that the greater increased by 6 may be to the less diminished by 11 as 9 to 2.

Let $x =$ the greater;

$$\therefore 49 - x = \text{the less},$$

$$\text{and } x + 6 : 38 - x :: 9 : 2;$$

$$\therefore (Alg.\ 179,\ 2\text{—}3.)\ x + 6 : 44 :: 9 : 11,$$

$$\text{and } (Alg.\ 179,\ 8.)\ x + 6 : 4 :: 9 : 1;$$

$$\therefore x + 6 = 36,$$

$$\text{and } x = 30;$$

$\therefore$ the parts are 30 and 19.

62. *A*, *B*, and *C* make a joint stock; *A* puts in £60 less than *B*, and £68 more than *C*; and the sum of the shares of *A* and *B* is to the sum of the shares of *B* and *C* as 5 to 4. What did each put in?

$$\text{Let } x = \text{what } A \text{ put in;}$$
$$\therefore x + 60 = \text{what } B \text{ put in,}$$
$$\text{and } x - 68 = \text{what } C \text{ put in;}$$
$$\text{then } 2x + 60 : 2x - 8 :: 5 : 4,$$
$$\text{and } (\textit{Alg.}\ 179,\ 7.)\ x + 30 : x - 4 :: 5 : 4;$$
$$\therefore (\textit{Alg.}\ 179,\ 4.)\ 34 : x - 4 :: 1 : 4;$$
$$\therefore 136 = x - 4,$$
$$\text{and } x = 140;$$

$\therefore$ they put in £140, £200, and £72 respectively.

63. It is required to divide the number 34 into two such parts, that the difference between the greater and 18, shall be to the difference between 18 and the less :: 2 : 3.

$$\text{Let } x = \text{the greater;}$$
$$\therefore 34 - x = \text{the less,}$$
$$\text{and } x - 18 : x - 16 :: 2 : 3;$$
$$\therefore (\textit{Alg.}\ 179,\ 2,\ 4.)\ x - 18 : 2 :: 2 : 1;$$
$$\therefore x - 18 = 4,$$
$$\text{and } x = 22;$$

$\therefore$ the parts are 22 and 12.

64. A Bookseller sells two books, one containing 100 sheets, for 10 shillings, the other containing 50 sheets, for 6 shillings, each being bound at the same price. What was that price?

$$\text{Let } x = \text{the price;}$$
$$\text{then } 10 - x : 6 - x :: 100 : 50 :: 2 : 1;$$
$$\therefore (\textit{Alg.}\ 179,\ 4.)\ 4 : 6 - x :: 1 : 1;$$
$$\therefore 4 = 6 - x;$$

by transposition, $x = 2$.

65. A man wished to inclose a piece of ground with palisades, and found that if he set them a foot asunder, he should have too few by 150; but if he set them a yard asunder, he should have too many by 70. How many had he?

Let $x =$ the number;

$$\text{then } x - 70 : x + 150 :: 1 : 3,$$

$$\text{and } (\textit{Alg.}\ 179,\ 5.)\ x - 70 : 220 :: 1 : 2,$$

$$\text{or } x - 70 : 110 :: 1 : 1;$$

$$\therefore x - 70 = 110,$$

$$\text{and } x = 180.$$

66. A Footman, who contracted for £8 a year, and a livery suit, was turned away at the end of 7 months, and received only £2. 3*s*. 4*d*. and his livery. What was its value?

Let $x =$ its value, in pounds;

$$\text{then } 12 : 7 :: \left(x + 8 : x + \frac{13}{6} ::\right) 6x + 48 : 6x + 13;$$

$$\therefore (\textit{Alg.}\ 179,\ 4.)\ 5 : 7 :: 35 : 6x + 13,$$

$$\text{and } 1 : 7 :: 7 : 6x + 13;$$

$$\therefore 6x + 13 = 49;$$

$$\text{by transposition, } 6x = 36,$$

$$\text{and } x = 6.$$

67. What number is that to which if 1, 5, and 13, be severally added, the first sum shall be to the second, as the second to the third?

Let $x =$ the number required;

$$\text{then } x + 1 : x + 5 :: x + 5 : x + 13;$$

$$\therefore (\textit{Alg.}\ 179,\ 5.)\ x + 1 : 4 :: x + 5 : 8,$$

$$\text{and alt}^{\text{do}}.\ x + 1 : x + 5 :: 4 : 8 :: 1 : 2;$$

$$\therefore x + 1 : 4 :: 1 : 1,$$

$$x + 1 = 4,$$

$$\text{and } x = 3.$$

68. A Landlord let his farm for £10 a year in money and a corn-rent. When corn sold at 10*s*. a bushel, he received at the rate of 10 shillings an acre for his land; but when it sold at 13*s*. 6*d*. a bushel, 13 shillings an acre. Of how many bushels did the corn-rent consist?

Let $x =$ the number of bushels;
then $10x + 200 =$ the annual income (in shillings);
and $\therefore\ x + 20 =$ the number of acres;

also in the second case $\dfrac{27x + 400}{26} =$ the number of acres;

$$\therefore\ \frac{27x + 400}{26} = x + 20,$$

and $27x + 400 = 26x + 520$,
by transposition, $x = 120$.

69. When the price of a bushel of barley wanted but 3*d.* to be to the price of a bushel of oats as 8 to 5, nine bushels of oats were received as an equivalent for four bushels of barley, and 7*s.* 6*d.* in money. What was the price of a bushel of each?

Let $5x =$ the price of a bushel of oats;
$\therefore\ 8x - 3 =$ the price of a bushel of barley;

$$45x = 32x - 12 + 90;$$

by transposition, $13x = 78$,
and $x = 6$;
$\therefore$ the price of a bushel of oats $=$ 30*d.*
and the price of a bushel of barley $=$ 45*d.*

70. A Countryman had two flocks of sheep, the smaller consisting entirely of ewes, each of which brought him 2 lambs. On counting them he found that the number of lambs was equal to the difference between the two flocks. If all his sheep had been ewes, and brought forth 3 lambs apiece, his stock would have been 432. Required the number in each flock.

Let $x =$ the number in the less;
$\therefore\ 2x =$ the number of lambs this flock produced $=$ the difference of the flocks,
and $3x =$ the number in the larger flock;

$$\therefore\ 4x + 3 \times 4x = 4 \times 4x = 432,$$

and $x = 27$;
$\therefore$ 27 and 81, are the numbers required.

71. A Market-woman bought a certain number of eggs at two a penny, and as many at three a penny, and sold them out at the rate of five for two-pence; after which she found that instead of making her money again, as she expected, she lost four-pence by them. How many eggs of each sort had she?

Let x = the number required;

then $2 : x :: 1 :$ the price of x eggs at 2 a penny $= \frac{x}{2}$;

in the same way, $\frac{x}{3}$ = the price of x eggs at 3 a penny.

and $5 : 2x :: 2 :$ the price at which she sold all, $= \frac{4x}{5}$;

$$\therefore \frac{4x}{5} + 4 = \frac{x}{2} + \frac{x}{3},$$

and $24x + 120 = (15x + 10x =)\ 25x$:

by transposition, $x = 120$.

72. A man and his wife did usually drink out a vessel of beer in 12 days: but when the man was out, the vessel lasted the woman 30 days. In how many days would the man alone drink it out?

Let x = the number of days required;

then $x : 12 :: 1 :$ part drunk by the man in 12 days $= \frac{12}{x}$,

and $30 : 12 :: 1 :$ part by the woman in 12 days $= \frac{12}{30} = \frac{2}{5}$;

$$\therefore \frac{12}{x} + \frac{2}{5} = 1,$$

and $60 + 2x = 5x$,

by transposition, $60 = 3x$,

and $20 = x$.

73. A cistern into which water was let by two cocks *A* and *B*, will be filled by them both running together in 12 hours, and by the cock *A* alone in 20 hours. In what time will it be filled by the cock *B* alone?

Let x = the number of hours;

then $x : 12 :: 1 :$ the quantity supplied by B in 12 hours $= \frac{12}{x}$.

In the same way, the quantity supplied by A in 12 hours $= \frac{3}{5}$;

$$\therefore \frac{12}{x} + \frac{3}{5} = 1,$$

$$\text{and } 60 + 3x = 5x;$$

$$\text{by transposition, } 60 = 2x,$$

$$\text{and } 30 = x.$$

74. The hold of a ship contained 442 gallons of water. This was emptied out by two buckets, the greater of which, holding twice as much as the other, was emptied twice in three minutes, but the less three times in two minutes; and the whole time of emptying was 12 minutes. Required the size of each.

Let x = the number of gallons the less held;

$\therefore 2x$ = the number the greater held;

and $4x$ = the quantity thrown out by the greater in 3 minutes;

$\therefore 3 : 12 :: 4x :$ the quantity thrown out in 12 minutes $= 16x$.

In the same manner the quantity thrown out by the less in 12 minutes $= 18x$;

$$\therefore (18x + 16x =)\ 34x = 442,$$

$$\text{and } x = 13;$$

$\therefore$ the less held 13, and the greater 26 gallons.

75. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's three; but two of the greyhound's leaps are as much as three of the hare's. How many leaps must the greyhound take to catch the hare?

Let $3x$ = the number of leaps the greyhound must take;

$\therefore 4x$ = the number the hare takes in the same time;

$\therefore 4x + 50$ = the whole number she takes,

$$\text{and } 2 : 3 :: 3x : 4x + 50;$$

$$\therefore 9x = 8x + 100;$$

$$\text{by transposition, } x = 100,$$

and the greyhound must take 300 leaps.

76. If 10 apples cost a penny, and 25 pears cost two-pence, and I buy 100 apples and pears for nine-pence halfpenny, how many of each shall I have?

Let $x =$ the number of apples;

$\therefore 100 - x =$ the number of pears;

and $10 : x :: 1 :$ the price of x apples $= \frac{x}{10}$.

In the same manner the price of the pears $= \frac{200 - 2x}{25}$;

$$\therefore \frac{x}{10} + \frac{200 - 2x}{25} = \frac{19}{2},$$

and $5x + 400 - 4x = 475$;

by transposition, $x = 75$;

$\therefore$ the number of apples is 75, and pears 25.

77. A person has two sorts of wine, one worth 20 pence a quart, and the other 12 pence; from which he would mix a quart to be worth 14 pence. How much of each must he take?

Let $x =$ the quantity of the first, the whole quart being represented by unity;

$\therefore 1 - x =$ the quantity of the second;

also $20x =$ the value of the first,

and $12 - 12x =$ the value of the second;

$\therefore 20x + 12 - 12x = 14$;

and by transposition, $8x = 2$;

$\therefore x = \frac{1}{4}$;

$\therefore$ he must take $\frac{1}{4}$ of the first, and $\frac{3}{4}$ of the second.

78. A person engaged to reap a field of 35 acres, consisting partly of wheat, and partly of rye. For every acre of rye he received 5 shillings; and what he received for an acre of wheat augmented by one shilling, is to what he received for an acre of rye as 7 to 3. For his whole labour he received £13. Required the number of acres of each sort.

Let $x =$ the number of acres of wheat;

$\therefore 35 - x =$ the number of acres of rye;

and $175 - 5x =$ the price of reaping them.

Now $3 : 7 :: 5 : 1 +$ the price of reaping an acre of wheat $= \frac{35}{3}$;

$\therefore$ the price of reaping an acre of wheat $= \frac{32}{3}$,

and the price of reaping all the wheat $= \frac{32x}{3}$;

$$\therefore \frac{32x}{3} + 175 - 5x = 260,$$

and $32x + 525 - 15x = 780$;
by transposition, $17x = 255$,
and $x = 15$;
$\therefore$ there were 15 acres of wheat, and 20 of rye.

79. Two pieces of cloth of equal goodness, but of different lengths, were bought, the one for £5, the other for £6. 10*s.* Now if the lengths of both pieces were increased by 10, the numbers resulting would be in the proportion of 5 to 6. How long was each piece, and how much did they cost a yard?

Let $x =$ the number of 10 shillings that each yard cost,

then $\frac{10}{x} =$ the length of the least,

and $\frac{13}{x} =$ the length of the longest;

$$\text{also } \frac{10}{x} + 10 : \frac{13}{x} + 10 :: 5 : 6;$$

$$\therefore (\textit{Alg.}\ 179,\ 5.)\ 10 \,.\left(\frac{1}{x} + 1\right) : \frac{3}{x} :: 5 : 1,$$

$$\text{and } (\textit{Alg.}\ 179,\ 8.)\ 2 \,.\left(\frac{1}{x} + 1\right) : \frac{3}{x} :: 1 : 1;$$

$$\therefore 2 \,.\frac{x+1}{x} = \frac{3}{x},$$

and $2x + 2 = 3$;
by transposition, $2x = 1$,
and $x = \frac{1}{2}$;

$\therefore$ the price is 5*s.* and the lengths are 20 and 26 yards.

80. A General, whose horse was $\frac{1}{3}$ of his foot, after a defeat found, that before the battle $\frac{1}{12} - 120$ of his foot, and $\frac{1}{12} + 120$ of his horse had deserted; $\frac{1}{4}$ of his whole army was in garrison; and $\frac{3}{8}$ remained, the rest being either taken prisoners or slain. Now $300 +$ the number slain $= \frac{1}{2}$ the foot he had at first. Of how many did his whole army consist?

Let $x =$ the number of horse;
$\therefore\ 3x =$ the number of foot,
and $4x =$ the whole army;
and $x =$ the number in garrison;

$$\text{also } \frac{3x}{2} - 300 = \text{the number slain;}$$

$$\text{and } \frac{3x}{2} = \text{the number that remained.}$$

$$\therefore\ \frac{x}{4} - 120 + \frac{x}{12} + 120 + x + \frac{3x}{2} + \frac{3x}{2} - 300 = 4x,$$

and $3x + x + 48x - 3600 = 48x$;
by transposition, $4x = 3600$,
and $x = 900$;

$\therefore$ the whole army consisted of 3600 men; viz. 900 horse, and 2700 foot.

81. Two persons *A* and *B* have both the same annual income. *A* lays by $\frac{1}{5}$th of his; but *B* by spending £80 per annum more than *A*, at the end of 4 years finds himself £220 in debt. What did each receive and expend annually?

Let $5x =$ their annual income;
$\therefore\ 4x =$ *A*'s annual expenditure,
and $4x + 80 =$ *B*'s annual expenditure;
$\therefore\ (4x + 80 - 5x =)\ 80 - x =$ the debt *B* annually incurs;
$\therefore\ 320 - 4x = 220$;
by transposition, $4x = 100$,
and $x = 25$;
$\therefore$ their annual income is £125;
A's annual expenditure is £100, and *B*'s £180.

82. A person at play won twice as much as he began with, and then lost 16 shillings. After this he lost four-fifths of what remained, and then won as much as he began with, and counting his money, found he had 80 shillings. What sum did he begin with?

Let $x =$ the number of shillings he began with;
then $3x =$ the sum he had, after winning $2x$,
and $3x - 16 =$ the sum remaining after the next loss.

Now since he lost $\frac{4}{5}$ of this, $\frac{3x-16}{5} =$ the sum remaining;

$$\therefore \frac{3x-16}{5} + x = 80,$$
and $3x - 16 + 5x = 400$;
by transposition, $8x = 416$,
and $x = 52$.

83. Having lost one-third of my money at play, I won 3 times as much as I had left, half as much money as I began with, and £50; and then found I had as much above £100, as the sum I began with was below £100. What sum did I begin with?

Let $6x =$ the number of pounds required;
then $4x =$ the sum remaining after $\frac{1}{3}$ was lost,
and $12x + 3x + 50 =$ the sum afterwards won;
$\therefore$ $(12x + 3x + 50 + 4x =)$ $19x + 50 =$ the whole sum he had,
and $19x + 50 - 100 = 100 - 6x$;
by transposition, $25x = 150$,
and $x = 6$;
$\therefore$ he began with 36 pounds.

84. *A* and *B* began to pay their debts. *A*'s money was at first two-thirds of *B*'s; but after *A* had paid £1 less than two-thirds of his money, and *B* £1 more than seven-eighths of his, it was found that *B* had only half as much as *A* had left. What sum had each at first?

Let $2x$ and $3x$ = the sums A and B had respectively,

$$\left.\begin{array}{r}\text{then after payment, } A \text{ had } \dfrac{2x}{3}+1 \\ \text{and } B \text{ had } \dfrac{3x}{8}-1\end{array}\right\} \text{remaining;}$$

$$\therefore \frac{2x}{3}+1=\frac{3x}{4}-2;$$

$$\therefore 8x+12=9x-24;$$

by transposition, $36 = x$;

$\therefore$ A had £72, and B had £108.

85. It is required to divide the number 36 into three such parts, that one-half of the first, one-third of the second, and one-fourth of the third may be equal to each other.

Let $2x$ = the first part;

$\therefore$ $x = \frac{1}{3}$ part of the second, and $3x$ = the second;

also $x = \frac{1}{4}$ part of the third, and $\therefore$ $4x$ = the third;

$$\therefore (2x+3x+4x=)\ 9x=36.$$

and $x = 4$;

$\therefore$ the parts are 8, 12, and 16.

86. Divide the number 116 into four such parts, that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the result in each case shall be the same.

Let x = the third;

$\therefore$ $3x$ = half the fourth,

and $6x$ = the fourth;

whence $3x + 4$ = the second,

and $3x - 5$ = the first;

$$\therefore 3x-5+3x+4+x+6x=116;$$

by transposition, $13x = 117$,

and $x = 9$;

$\therefore$ the parts are 22, 31, 9, and 54.

87. A Gentleman had some of his horses at grass at 3 shillings each a week, and the rest at livery stables at 10 shillings each a week. The horses in the stables cost him twice as

much a week as the horses at grass. But he finds that if he had sent 3 horses to grass out of the stables, the expense of the stables would have been only 6 shillings a week more than the grass. How many horses had he?

Let $x =$ the number of horses at grass;

$\therefore\ 3x =$ the weekly expense of these,

and $6x =$ - - - - - - - - - - - of horses in stables;

$$\therefore\ \frac{6x}{10} = \text{their number};$$

also $6x - 30 =$ the expense of the stables after 3 horses were sent out to grass,

and $3x + 9 =$ the expense of the horses at grass, when 3 more were added;

$$\therefore\ 6x - 30\ (= 3x + 9 + 6) = 3x + 15;$$

by transposition, $3x = 45$;

and $x = 15$;

$\therefore$ there were 15 at grass, and 9 in the stables.

88. A Silversmith received in payment for a certain weight of wrought plate, the price of which was £10, the same weight of unwrought plate, and £3. 15*s*. besides. At another time he exchanged 12 oz. of wrought plate of the same workmanship as before for 8 oz. of unwrought (for which he allowed the same price as before), and £2. 16*s*. in money. What was the price of wrought plate per ounce, and the weight of the first sold?

Let $x =$ the number of ounces;

$$\therefore\ \frac{200}{x} = \text{the price of an oz. wrought,}$$

$$\text{and } \frac{125}{x} = \text{the price of an oz. unwrought;}$$

$$\therefore\ \frac{2400}{x} = \frac{1000}{x} + 56;$$

$$\text{by transposition, } \frac{1400}{x} = 56.$$

$$\text{and } \frac{25}{x} = 1;$$

$$\therefore\ x = 25;$$

whence there were 25 ounces; and the price was 8 shillings per oz.

89. In changing a bill of £85 into guineas and shillings (the number of shillings being $\frac{1}{4}$ number of guineas) on examination they all proved adulterated below the standard value, to the amount in the whole of £8. 5*s*. To make up the deficiency, nine more such guineas were paid; and four such shillings and three good ones returned. Required the number and average value of the guineas and shillings paid at first.

£85 $= 85s. \times 20 =$ the N°. of shillings $+ 4 \times 21 \times$ N°. of shillings

$= 85 \times$ the number;

$\therefore$ the number of shillings $= 20$,

and the number of guineas $= 80$;

Let $x =$ the value of an adulterated guinea;

$\therefore$ $(1700 - 165 =)$ $1535 = 80x + 20 \times$ value of an adulterated shilling,

and $\therefore$ the value of an adulterated shilling $= \dfrac{1535 - 80x}{20} = \dfrac{307 - 16x}{4}$;

$$\therefore 165 = 9x - 3 - 307 + 16x = 25x - 310;$$

by transposition, $475 = 25x$,

and $19 = x$;

$\therefore$ the value of an adulterated guinea $= 19s.$ and the value of an adulterated shilling $= \dfrac{307 - 16x}{4} = \frac{3}{4} = 9d.$

90. Before noon, a clock which is too fast, and points to afternoon time, is put back five hours and forty minutes; and it is observed that the time before shown is to the true time as 29 to 105. Required the true time.

Let $x =$ the time the clock pointed to;

then $x : x + 6\frac{1}{3} :: 29 : 105$;

(*Alg.* 179, 5.) $x : \dfrac{19}{3} :: 29 : 76$,

and $x : \dfrac{1}{3} :: 29 : 4$;

$$\text{whence } x = \frac{29}{12} = 2^{\text{h}}\,25';$$

if $\therefore$ this be added to $6^{\text{h}}\,20'$, the true time is $8^{\text{h}}\,45'$, or $15'$ before 9.

91. The crew of a ship consisted of her complement of sailors and a number of soldiers. Now there were 22 seamen to every 3 guns and 10 over. Also the whole number of hands was 5 times the number of soldiers and guns together. But after an engagement, in which the slain were one-fourth of the survivors, there wanted 5, to be 13 men to every 2 guns. Required the number of guns, soldiers, and sailors.

Let $3x$ = the number of guns;

then $22x + 10$ = the number of seamen,

and since, seamen + soldiers = 5 . soldiers + $15x$;

$$\therefore \text{ the number of soldiers} = \tfrac{1}{4}\,(22x + 10 - 15x) = \frac{7x + 10}{4},$$

$$\text{and the complement} = \frac{7x + 10}{4} + 22x + 10 = \frac{95x + 50}{4},$$

$$\text{and the survivors were } \tfrac{4}{5}\left(\frac{95x + 50}{4}\right) = 19x + 10;$$

$$\therefore \frac{3x}{2} \,.\, 13 - 5 = 19x + 10;$$

$$\text{by transposition, } \frac{x}{2} = 15,$$

$$\text{and } x = 30;$$

$\therefore$ the number of guns = 90;

the number of seamen = $30 \times 22 + 10 = 670$,

$$\text{and the number of soldiers} = \frac{7 \times 30 + 10}{4} = 55.$$

92. A Shepherd, in time of war, was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock, and $\frac{1}{4}$ of a sheep; another party took from him $\frac{1}{3}$ of what he had left, and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what now remained, and $\frac{1}{2}$ of a sheep. After which he had but 25 sheep left. How many had he at first?

Let x = the number he had at first;

$$\text{then } \frac{x+1}{4} = \text{the number the first party took away,}$$

$$\text{and } \therefore \frac{3x-1}{4} = \text{the number remaining.}$$

Now the second party took away $\frac{1}{3}$ of these + $\frac{1}{3}$ of a sheep; $\therefore$ there remained

$$\tfrac{2}{3}\left(\frac{3x-1}{4}\right) - \tfrac{1}{3} = \frac{3x-1}{6} - \tfrac{1}{3} = \frac{3x-3}{6} = \frac{x-1}{2};$$

then the third party took away half of these + $\frac{1}{2}$ of a sheep;

$$\therefore \text{there remained } \frac{x-1}{4} - \tfrac{1}{2} = \frac{x-3}{4};$$

$$\text{whence } \frac{x-3}{4} = 25,$$

$$\text{and } x - 3 = 100;$$

$$\text{by transposition, } x = 103.$$

93. A man being at play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had, and won 2 shillings; lastly he lost $\frac{1}{7}$ of what he then had; this done he had but 12 shillings left. What had he at first?

Let $4x$ = the number of shillings required;

then after the first loss he had $3x$, and afterwards $3x + 3$; after the second loss he had $\frac{2}{3} \cdot (3x + 3) = 2x + 2$, and afterwards $2x + 4$.

Having lost $\frac{1}{7}$ of this, he had $\frac{6}{7} \cdot (2x + 4) = \frac{12x + 24}{7}$,

$$\text{and } \therefore \frac{12x + 24}{7} = 12,$$

$$\text{and } 12x + 24 = 7 \times 12;$$

$$\text{by transposition, } 12x = 5 \times 12,$$

$$\text{and } x = 5;$$

$\therefore$ he had at first 20 shillings.

94. A Trader maintained himself for 3 years at the expense of £50 a year; and in each of those years augmented that part of his stock which was not so expended by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled. What was that stock?

Let $x =$ the number of pounds required;
then $x - 50 =$ the sum not expended; and with this he traded;

$\therefore \dfrac{x-50}{3} =$ his gain the first year,

and $\dfrac{4}{3} \cdot (x - 50) =$ the sum he had at the end of the first year;

$\therefore \dfrac{4x-200}{3} - 50 = \dfrac{4x-350}{3} =$ the sum he traded with the second year;

$\therefore \dfrac{4}{3} \cdot \dfrac{4x-350}{3} = \dfrac{16x-1400}{9} =$ the sum he had at the end of the second year;

and $\dfrac{16x-1400}{9} - 50 = \dfrac{16x-1850}{9} =$ the sum he traded with the third year;

$\therefore \dfrac{4}{3} \cdot \dfrac{16x-1850}{9} =$ the sum he had at the end of the third year;

$$\text{whence } \frac{4}{3} \cdot \frac{16x-1850}{9} = 2x,$$
$$\text{and } 32x - 3700 = 27x;$$
$$\text{by transposition, } 5x = 3700,$$
$$\text{and } x = 740.$$

95. A Merchant buys a cask of brandy for £48, and sells a quantity exceeding three-fourths of the whole by 2 gallons at a profit of £25 per cent. He afterwards sells the remainder at such a price as to clear £60 per cent. by the whole transaction; and, had he sold the whole quantity at the latter price, he would have gained £175 per cent. Required the number of gallons contained in the cask.

Let $4x =$ the number of gallons;

then $\dfrac{12}{x} =$ the original price per gallon (in pounds),

and $100 : 125 :: \dfrac{12}{x} :$ the first price of sale $= \dfrac{15}{x}$,

$$\text{and } 100 : 275 :: \frac{12}{x} : \text{the latter price of sale} = \frac{33}{x};$$

$$\therefore\ (3x + 2) \cdot \frac{15}{x} + (x - 2) \cdot \frac{33}{x} - 48 = \text{whole gain} = 30 - \frac{36}{x};$$

$$\text{hence } 100 : 60 :: 48 : 30 - \frac{36}{x},$$

$$\text{or } 5 : 3 :: 8 : 5 - \frac{6}{x},$$

$$\therefore\ 24 = 25 - \frac{30}{x};$$

$$\text{by transposition, } \frac{30}{x} = 1,$$

$$\text{and } \therefore\ x = 30,$$

and the number required $= 4x = 120$ gallons.

96. Water flows uniformly into a cistern, capable of containing 720 gallons, through a pipe; and at the same time is discharged by a pump, worked by three men, who take four strokes in a minute; but this not being sufficient, the cistern becomes full in 6 hours; they therefore now put in another pump, of such power that the quantity discharged at one stroke by this pump is to the quantity discharged at one stroke by the former :: 2 : 3; but being obliged to detach one of their number to work the pump, the former pump makes only 10 strokes in 3 minutes, and the latter 5 strokes in two minutes; by which means the cistern is emptied in 12 hours. How much water was discharged by each pump at one stroke? and how much flowed in through the pipe in one minute?

Let $3x =$ the number of gallons discharged by the first pump at one stroke;

$\therefore\ 2x =$ the number discharged by the second,

and $12x =$ the quantity discharged by the first in one minute, when 3 men worked;

$\therefore\ 6 \times 60 \times 12x =$ the quantity discharged in six hours;

$\therefore\ 6 \times 60 \times 12x + 720 = 720 \cdot (6x + 1) =$ the quantity introduced through the pipe in that time.

Now when the additional pump is worked,

$10x =$ quantity discharged by the first in one minute;

and $5x =$ the quantity discharged by the second in one minute;

$\therefore\ 15x \times 12 \times 60 =$ the whole quantity discharged in 12 hours;

$$\therefore\ 15x \times 720 = 720 \times 2 \,.\, (6x + 1) + 720;$$

$$\text{or } 15x = 12x + 3;$$

by transposition, $3x = 3$,

and $x = 1$;

$\therefore$ the first discharged 3 gallons, and the second 2, at one stroke; and the quantity introduced by the pipe in one minute $= \dfrac{720 \times (6x + 1)}{6 \times 60} = 2 \times 7 = 14$ gallons.

97. A poor man with a wife and seven children, found during a scarcity that he could only earn sufficient to procure $\frac{1}{4}$ of a white loaf of bread *per* day for each of his family, himself included. He therefore applied to the parish-officers for assistance, by whom being allowed a daily sum $= \frac{1}{2}$ his earnings, and mixed bread being made by order of Parliament, which was cheaper than white in the proportion of 4 to 5, he was now enabled to procure $\frac{1}{3}$ of a mixed loaf *per* day for each of the family (himself still included) and had 1*s*. $7\frac{1}{2}$*d*. over. Required the sum allowed him by the parish.

Let $x =$ the price of a white loaf (in pence);

$$\therefore\ \frac{9x}{4} = \text{what he earned},$$

$$\text{and } \frac{9x}{8} = \text{what the parish allowed him};$$

$$\text{also } \frac{4x}{5} = \text{the price of a mixed loaf};$$

$$\therefore\ \frac{12x}{5} + \frac{39}{2} = \left(\frac{9x}{4} + \frac{9x}{8} =\right)\frac{27x}{8},$$

$$\text{and } 96x + 780 = 135x;$$

by transposition, $780 = 39x$,

and $20 = x$;

whence it appears that he earned 45 pence, and had $22\frac{1}{2}$*d*. allowed him by the parish.

SECTION VII.

Examples of the Solution of Problems producing Simple Equations involving two unknown Quantities.

1. AFTER A had won four shillings of B, he had only half as many shillings as B had left. But had B won six shillings of A, then he would have had three times as many as A would have had left. How many had each?

Let $x =$ the number of shillings A had,
and $y =$ the number B had;

$$\text{then } y - 4 = 2x + 8,$$
$$\text{and } y + 6 = 3x - 18;$$
$$\therefore \text{ by subtraction, } 10 = x - 26,$$
$$\text{and by transposition, } 36 = x,$$
and $y - 4 = 80$; $\therefore$ $y = 84$; $\therefore$ A had 36, and B 84.

2. A person bought a quantity of brandy and rum for £19. 4*s*., and gave for the brandy 9 shillings, and rum 6 shillings *per* bottle. He found however that he could have bought as many bottles of rum as he now had of brandy, and as many of brandy as he now had of rum for £1. 13*s*. less. How much was bought?

Let $x =$ the number of bottles of brandy,
and $y =$ the number of bottles of rum;

$$\text{then } 9x + 6y = 384,$$
$$\text{and } 6x + 9y = 351;$$
$$\therefore \text{ by addition, } 15x + 15y = 735,$$
$$\text{and } x + y = 49.$$
$$\text{But since } 3x + 2y = 128,$$
$$\text{and } 2x + 2y = 98;$$
$$\therefore \text{ by subtraction, } x = 30,$$
$$\text{and } y = 49 - x = 19;$$

$\therefore$ he bought 30 bottles of brandy, and 19 of rum.

3. What fraction is that, to the numerator of which if 4 be added, the value is one-half, but if 7 be added to the denominator, its value is one-fifth?

$$\text{Let } \frac{x}{y} = \text{the fraction required;}$$
$$\text{then } \frac{x+4}{y} = \frac{1}{2}, \text{ and } \therefore\ 2x + 8 = y;$$
$$\text{also } \frac{x}{y+7} = \frac{1}{5}, \text{ and } \therefore\ 5x = y + 7;$$
$$\text{by subtraction, } 3x - 8 = 7;$$
$$\text{by transposition, } 3x = 15;$$
$$\text{and } x = 5;$$
$$\therefore\ y = 2x + 8 = 18,$$
$$\text{and the fraction is } \frac{5}{18}.$$

4. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.

$$\text{Let } x \text{ and } y = \text{the numbers;}$$
$$\text{then } x : y :: x + y : 42,$$
$$\text{and } x : y :: x - y : 6.$$

But ratios which are equal to the same ratio are equal to each other;

$$\therefore\ x + y : 42 :: x - y : 6,$$
$$\text{alt}^{\text{do}}.\ x + y : x - y :: (42 : 6 ::)\ 7 : 1;$$
$$\therefore\ (\textit{Alg. } 189,\ 6.)\ 2x : 2y :: 8 : 6,$$
$$\text{and } x : y :: 4 : 3;$$
$$\therefore\ x = \frac{4y}{3},$$
$$\text{and } 4 : 3 :: \frac{y}{3} : 6;$$
$$\therefore\ y = 24,$$
$$\text{and } x = \frac{4y}{3} = 32;$$
$$\therefore \text{ the numbers are } 32 \text{ and } 24.$$

5. What two numbers are those, whose difference, sum, and product, are as the numbers 2, 3, and 5, respectively?

Let x and $y =$ the numbers;

then $x - y : x + y :: 2 : 3$;

$\therefore 2x : 2y :: 5 : 1$,

or $x : y :: 5 : 1$;

also $x + y : xy :: 3 : 5$,

or $6y : xy :: 3 : 5$;

$\therefore 2 : x :: 1 : 5$,

and $\therefore x = 10$,

and $y = \frac{x}{5} = 2$;

$\therefore$ the numbers are 10 and 2.

6. *A* and *B* playing at bowls, says *A* to *B*, If you will give me a guinea, I will bet you half a crown to eighteen pence on each game, and will play 36 games together. *B* won his guinea back again, and £1. 17*s.* besides. How many games did each win?

Let $x =$ the number of games *A* won,

and $y =$ the number *B* won;

$\therefore y + x = 36$,

and $3y + 3x = 108$;

but $5y - 3x = 116$;

$\therefore$ by addition, $8y = 224$,

and $y = 28$;

$\therefore x = 36 - y = 8$;

$\therefore$ *A* won 8, and *B* 28 games.

7. A person exchanged 12 bushels of wheat for 8 bushels of barley, and £2. 16*s.*; offering at the same time to sell a certain quantity of wheat for an equal quantity of barley, and £3. 15*s.* in money, or for £10 in money. Required the prices of the wheat and barley per bushel.

Let $x =$ the price of wheat per bushel, in shillings,

and $y =$ the price of barley;

then $\frac{200}{x} =$ the number of bushels in the second offer;

$$\therefore\ 12x = 8y + 56,$$
$$\text{and } \frac{200}{x} \times y = 125;$$
$$\therefore\ 8y = 5x,$$
$$\text{and } \therefore\ 12x = 5x + 56;$$
$$\text{by transposition, } 7x = 56,$$
$$\text{and } x = 8;$$
$$\therefore\ y = 5;$$

∴ the prices of wheat and barley *per* bushel were 8 and 5 shillings, respectively.

8. A Vintner sold at one time 20 dozen of port wine, and 30 of sherry, and for the whole received £120; and at another time sold 30 dozen of port, and 25 of sherry at the same prices as before, and for the whole received £140. What was the price of a dozen of each sort of wine?

Let $x =$ the price of a dozen of port,
and $y =$ - - - - - - - - - of sherry;

$$\therefore\ 20x + 30y = 120, \text{ or } 2x + 3y = 12,$$
$$\text{and } 30x + 25y = 140, \text{ or } 6x + 5y = 28.$$

Multiplying the first equation by 3,

$$6x + 9y = 36,$$
$$\text{but } 6x + 5y = 28;$$
$$\therefore \text{ by subtraction, } 4y = 8,$$
$$\text{and } y = 2;$$
$$\text{whence } 2x = 12 - 3y = 12 - 6 = 6,$$
$$\text{and } x = 3;$$

∴ the prices of port and sherry *per* dozen were £3 and £2, respectively.

9. *A* and *B* severally cut packs of cards, so as to cut off less than they left. Now the number of cards left by *A* added to the number cut off by *B* make 50; also the number of cards left by both exceed the number cut off, by 64. How many did each cut off?

Let $x =$ the number cut off by A;
$\therefore\ 52 - x =$ the number left by him.

$$\text{Let } y = \text{the number cut off by } B;$$
$$\therefore\ 52 - y = \text{the number left by him};$$
$$\text{then } x + y = \text{the whole number cut off,}$$
$$\text{and } 104 - (x + y) = \text{the whole number left,}$$
$$\text{whence } 104 - 2 \,.\, (x + y) = 64;$$
$$\text{by transposition, } 2 \,.\, (x + y) = 40,$$
$$\text{and } x + y = 20.$$
$$\text{Now } 52 - x + y = 50;$$
$$\therefore \text{ by transposition, } x - y = 2,$$
$$\text{but } x + y = 20;$$
$$\therefore \text{ by addition, } 2x = 22,$$
$$\text{and } x = 11;$$
$$\text{by subtraction, } 2y = 18,$$
$$\text{and } y = 9;$$
$$\therefore\ A \text{ cut off } 11, \text{ and } B\ 9.$$

10. A countryman, being employed by a poulterer to drive a flock of geese and turkeys to London, in order to distinguish his own from any he might meet on the road, pulled 3 feathers out of the tail of each turkey, and one out of the tail of each goose, and upon counting them, found that the number of turkeys' feathers exceeded twice those of the geese by 15. Having bought 10 geese and sold 15 turkeys by the way, he was surprised to find, as he drove them into the poulterer's yard, that the number of geese exceeded the number of turkeys in the proportion of 7 to 3. Required the number of each at first.

Let $x =$ the number of turkeys;

$\therefore\ 3x =$ the number of feathers from their tails;

let $y =$ the number of geese; and $\therefore$ of the feathers,

$$\text{and } 3x - 2y = 15.$$
$$\text{Also } y + 10 : x - 15 :: 7 : 3;$$
$$\therefore\ 3y + 30 = 7x - 105;$$
$$\text{by transposition, } 7x - 3y = 135,$$
$$\text{and } 14x - 6y = 270;$$
$$\text{but from the first equation, } 9x - 6y = 45;$$

$$\therefore \text{ by subtraction, } 5x = 225,$$
$$\text{and } x = 45;$$
$$\therefore 2y = 3x - 15 = 135 - 15 = 120,$$
$$\text{and } y = 60;$$
$\therefore$ there were 45 turkeys, and 60 geese.

11. A Farmer with 28 bushels of barley at 2*s.* 4*d.* a bushel, would mix rye at 3 shillings *per* bushel, and wheat at 4 shillings *per* bushel, so that the whole mixture may consist of 100 bushels, and be worth 3*s.* 4*d.* *per* bushel. How many bushels of rye, and how many of wheat, must he mix with the barley?

Let x = the number of bushels of rye,
and y = the number of wheat;
then the value of the barley = 196 (fourpences),
of the wheat $= 12y$,
of the rye $= 9x$;

$$\therefore 196 + 9x + 12y = 1000;$$
$$\text{by transposition, } 9x + 12y = 804,$$
$$\text{and } 3x + 4y = 268.$$
$$\text{Now } x + y + 28 = 100,$$
$$\text{and by transposition, } x + y = 72;$$
$$\therefore 3x + 3y = 216,$$
$$\text{but } 3x + 4y = 268;$$
$$\therefore \text{ by subtraction, } y = 52,$$
$$\text{and } x = 72 - y = 20.$$

Hence he must mix 20 bushels of rye, and 52 of wheat.

12. *A* and *B* speculate with different sums; *A* gains £150, *B* loses £50, and now *A*'s stock is to *B*'s as 3 to 2. But had *A* lost £50, and *B* gained £100, then *A*'s stock would have been to *B*'s as 5 to 9. What was the stock of each?

Let x = *A*'s stock,
and y = *B*'s;

$$\text{then } x + 150 : y - 50 :: 3 : 2;$$
$$\therefore 2x + 300 = 3y - 150,$$
$$\text{and by transposition, } 3y - 2x = 450;$$

$$\text{also } x - 50 : y + 100 :: 5 : 9;$$
$$\therefore\ 9x - 450 = 5y + 500;$$
$$\text{by transposition, } 9x - 5y = 950;$$

multiplying this equation by 3, and that found above by 5,

$$27x - 15y = 2850,$$
$$\text{and } 15y - 10x = 2250;$$
$$\therefore \text{ by addition, } 17x = 5100,$$
$$\text{and } x = 300;$$
$$\therefore\ 3y = 2x + 450 = 1050,$$
$$\text{and } y = 350;$$

∴ *A*'s was £300, and *B*'s £350.

13. A Merchant having mixed a certain number of gallons of brandy and water, found that if he had mixed 6 gallons more of each, he would have put into the mixture 7 gallons of brandy for every 6 of water; but if he had mixed 6 less of each, he would have put in 6 gallons of brandy for every 5 of water. How many of each did he mix?

Let x = the number of gallons of brandy,
and y = the number of gallons of water;

$$\text{then } x + 6 : y + 6 :: 7 : 6;$$
$$\therefore\ (\textit{Alg. } 179, 5.)\ x + 6 : x - y :: 7 : 1,$$
$$\text{but } x - 6 : y - 6 :: 6 : 5,$$
$$\text{and } \therefore\ x - y : x - 6 :: 1 : 6,$$
$$\text{and since } x + 6 : x - y :: 7 : 1;$$
$$\therefore\ \textit{ex aequali},\ x + 6 : x - 6 :: 7 : 6,$$
$$\text{and } 2x : 12 :: 13 : 1,$$
$$\text{or } x : 6 :: 13 : 1;$$
$$\therefore\ x = 78;$$
$$\text{whence } 84 : y + 6 :: 7 : 6,$$
$$\text{or } 12 : y + 6 :: 1 : 6;$$
$$\therefore\ y + 6 = 72,$$
$$\text{and } y = 66;$$

∴ he mixed 78 gallons of brandy with 66 of water.

14. A person had a bag of money worth £93; but a servant having robbed him of one-sixth of his moidores, and three-

fifths of his guineas, left him only £54. 15s. How many moidores and guineas had he at first?

Let $x =$ the number of guineas,
and $y =$ the number of moidores;

$$\text{then } \frac{45y}{6} + \frac{14x}{5} = 365,$$

$$\text{and } \therefore\ 75y + 28x = 3650;$$

$$\text{also } 7x + 9y = 620;$$

$$\therefore\ 36y + 28x = 2480;$$

$$\text{but since } 75y + 28x = 3650;$$

$$\therefore \text{ by subtraction, } 39y = 1170,$$

$$\text{and } y = 30;$$

$$\text{also } 7x = 620 - 9y = 620 - 270 = 350;$$

$$\therefore\ x = 50;$$

$\therefore$ he had 50 guineas, and 30 moidores.

15. A Vintner bought 6 dozen of port wine and 3 dozen of white for 12 guineas; but the price of each afterwards falling a shilling *per* bottle, he had 20 bottles of port, and 3 dozen and 8 bottles of white more, for the same sum. What was the price of each at first?

$$\left.\begin{array}{l}\text{Let } x = \text{the price of the port}\\ \quad y = \text{- - - - - - - - - white}\end{array}\right\} \textit{per} \text{ bottle (in shillings)},$$

$$\text{then } 72x + 36y = 252 = 92x + 80y - 172,$$

$$\text{and } \therefore \text{ by transposition, } 20x + 44y = 172,$$

$$\text{or } 5x + 11y = 43.$$

$$\text{Now, since } 72x + 36y = 252;$$

$$\therefore\ 2x + y = 7,$$

$$\text{and } 22x + 11y = 77,$$

$$\text{but } 5x + 11y = 43;$$

$$\therefore \text{ by subtraction, } 17x = 34,$$

$$\text{and } x = 2;$$

$$\text{whence } y = 7 - 2x = 7 - 4 = 3;$$

$\therefore$ the price of port was 2s. and of white 3s. *per* bottle.

16. A rectangular bowling-green having been measured, it was observed, that if it were 5 yards broader, and 4 yards longer,

it would contain 116 yards more: but if it were 4 yards broader, and 5 yards longer, it would contain 113 yards more. Required the length and breadth.

Let $x =$ the number of yards in length,

and $y =$ the number of yards in breadth;

then $(x + 4) \cdot (y + 5) = xy + 5x + 4y + 20 = 116 + xy$,

and $\therefore\ 5x + 4y = 96$;

also $(x + 5) \cdot (y + 4) = xy + 4x + 5y + 20 = 113 + xy$;

$\therefore\ 4x + 5y = 93$;

multiplying the former equation by 4, and the latter by 5,

$$20x + 16y = 384,$$

and $20x + 25y = 465$;

by subtraction, $9y = 81$,

and $y = 9$;

$\therefore\ 4x = 93 - 5y = 93 - 45 = 48$,

and $x = 12$;

$\therefore$ the length was 12, and the breadth 9 yards.

17. Find two numbers in the proportion of 5 to 7, to which two other required numbers in the proportion of 3 to 5 being respectively added, the sums shall be in the proportion of 9 to 13; and the difference of those sums $= 16$.

Let $5x$ and $7x =$ the two first numbers,

and $3y$ and $5y =$ the others;

then $5x + 3y : 7x + 5y :: 9 : 13$;

$\therefore\ 5x + 3y : 2x + 2y :: 9 : 4$;

or $5x + 3y : x + y :: 9 : 2$,

and $10x + 6y = 9x + 9y$;

by transposition, $x = 3y$;

but $2x + 2y = 16$;

$\therefore\ (6y + 2y =)\ 8y = 16$;

$\therefore\ y = 2$,

and $x = 6$;

whence, the two first numbers are 30 and 42; the two others, 6 and 10.

18. A Merchant finds that if he mixes sherry and brandy in quantities which are in the proportion of 2 to 1, he can sell

the mixture at 78 shillings a dozen; but if the proportion be as 7 to 2, he must sell it at 79 shillings a dozen. Required the price of each liquor.

$$\left.\begin{aligned}\text{Let } x &= \text{the price of the sherry}\\ y &= \text{the price of the brandy}\end{aligned}\right\}\textit{per}\text{ dozen;}$$

$$\text{then } 2x + y = 3 \times 78 = 234,$$
$$\text{and } 7x + 2y = 9 \times 79 = 711;$$

$$\left.\begin{array}{l}\text{but the first equation}\\ \text{being multiplied by } 2,\end{array}\right\}\; 4x + 2y = 468;$$

$$\therefore \text{ by subtraction, } 3x = 243,$$
$$\text{and } x = 81;$$

whence, $y = 234 - 2x = 234 - 162 = 72$;

$\therefore$ the price of the sherry was 81*s.*, and of the brandy 72*s.*

19. A Corn-factor mixes wheat-flour, which costs him 10 shillings a bushel, with barley-flour, which costs him 4 shillings a bushel, in such proportion, as to gain $43\frac{3}{4}$ *per* cent., by selling the mixture at 11 shillings a bushel. Required the proportion.

Let the proportion be $x : y$;

then $10x + 4y =$ the cost of $x + y$ bushels,

and $11x + 11y =$ the selling price;

$\therefore\ x + 7y =$ the gain;

whence,

$$10x + 4y : x + 7y :: 100 : 43\tfrac{3}{4} :: 400 : 175 :: 16 : 7,$$
$$\text{and } 5x + 2y : x + 7y :: 8 : 7;$$
$$\therefore 35x + 14y = 8x + 56y;$$
$$\text{by transposition, } 27x = 42y,$$
$$\text{and } 9x = 14y;$$
$$\therefore x : y :: 14 : 9,$$

and $\therefore$ he must mix 14 bushels of wheat-flour with 9 of barley.

20. A number consisting of 2 digits when divided by 4, gives a certain quotient and a remainder of 3; when divided by 9 gives another quotient and a remainder of 8. Now the *value* of the digit on the left hand is equal the quotient which was got when the number was divided by 9; and

the other digit is equal $\frac{1}{17}$th of the quotient got when the number was divided by 4. Required the number.

Let x and y = the digits in order;

then $10x + y$ = the number,

$$\text{and } \frac{10x + y}{9} = x + \frac{8}{9};$$

$$\therefore\ 10x + y = 9x + 8;$$

by transposition, $x + y = 8$;

$$\text{also } \frac{10x + y}{4} = \frac{3}{4} + 17y;$$

$$\therefore\ 10x + y = 3 + 68y;$$

by transposition, $10x - 67y = 3$;

but from the preceding equation, $10x + 10y = 80$;

$\therefore$ by subtraction, $77y = 77$,

and $y = 1$;

$\therefore\ x = 8 - y = 7$,

and the number required is 71.

21. A man and his wife could drink a barrel of beer in 15 days. After drinking together 6 days, the woman alone drank the remainder in 30 days. In what time would either alone drink a barrel?

Let x = the number of days in which the man could drink it,

and y = the number in which the woman could drink it;

$$\text{then } \frac{15}{x} + \frac{15}{y} = 1,$$

$$\text{and } \frac{6}{x} + \frac{6}{y} + \frac{30}{y} = 1,$$

$$\text{or } \frac{6}{x} + \frac{36}{y} = 1;$$

$$\text{hence from the first equation, } \frac{1}{x} + \frac{1}{y} = \frac{1}{15},$$

$$\text{from the last, } \frac{1}{x} + \frac{6}{y} = \frac{1}{6},$$

$$\therefore \text{ by subtraction, } \frac{5}{y} = \frac{1}{6} - \frac{1}{15} = \frac{3}{30} = \frac{1}{10},$$

$$\text{and } \therefore y = 50;$$

$$\text{also } \frac{1}{x} = \frac{1}{15} - \frac{1}{y} = \frac{1}{15} - \frac{1}{50} = \frac{7}{150};$$

$$\text{and } \therefore x = 21\tfrac{3}{7};$$

$\therefore$ the man would drink it in $21\frac{3}{7}$ days, and the woman in 50 days.

22. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many will it hold of each?

Let x = the number of crowns,
and y = the number of guineas;

then $x : 1 :: 4 :$ the space occupied by 4 crowns $= \frac{4}{x}$.

In the same way, the space occupied by 5 guineas $= \frac{5}{y}$;

$$\therefore \frac{4}{x} + \frac{5}{y} = \frac{17}{63},$$

$$\text{and } \frac{19}{x} + \frac{6}{y} = 1.$$

The first equation being multiplied by 6, and the second by 5,

$$\frac{24}{x} + \frac{30}{y} = \frac{34}{21},$$

$$\text{and } \frac{95}{x} + \frac{30}{y} = 5;$$

$$\therefore \text{ by subtraction, } \frac{71}{x} = 5 - \frac{34}{21} = \frac{71}{21};$$

$$\therefore x = 21,$$

$$\text{and } \frac{5}{y} = \frac{17}{63} - \frac{4}{21} = \frac{5}{63};$$

$$\therefore y = 63;$$

$\therefore$ the purse would hold 21 crowns, or 63 guineas.

23. Some smugglers discovered a cave, which would exactly hold the cargo of their boat, viz. 13 bales of cotton, and 33 casks of rum. Whilst they were unloading, a custom-house cutter coming in sight, they sailed away with 9 casks and five bales, leaving the cave two-thirds full. How many bales or casks would it hold?

Let x = the number of bales,
and y = the number of casks;

$$\text{then } \frac{13}{x} + \frac{33}{y} = 1,$$

$$\text{and } \frac{5}{x} + \frac{9}{y} = \frac{1}{3}.$$

Multiplying the first equation by 3, and the second by 11,

$$\text{then } \frac{39}{x} + \frac{99}{y} = 3,$$

$$\text{and } \frac{55}{x} + \frac{99}{y} = \frac{11}{3};$$

$$\therefore \text{ by subtraction, } \frac{16}{x} = \frac{2}{3};$$

$$\therefore 2x = 48,$$

$$\text{and } x = 24;$$

$$\text{consequently } \frac{9}{y} = \frac{1}{3} - \frac{5}{x} = \frac{1}{3} - \frac{5}{24} = \frac{3}{24} = \frac{1}{8};$$

$$\text{and } y = 72;$$

and $\therefore$ the cave would hold 24 bales, or 72 casks.

24. Round two wheels, whose circumferences are as 5 to 3, two ropes are wrapped, whose difference exceeds the difference of the circumferences by 280 yards. Now the larger rope applied to the larger wheel wraps round it a certain number of times, greater by 12 than the smaller round the smaller wheel; and if the larger wheel turns round 3 times as quick as the other, the ropes will be discharged at the same time. Required the lengths of the ropes and the circumferences of the wheels.

Let $5x$ and $3x$ = the circumferences of the wheels (in yards);

then $2x + 280$ = the difference of the ropes,

and $(15x : 3x ::)\ 5 : 1 ::$ the length of the longer string : the length of the shorter.

Let $\therefore\ 5y$ and y = the lengths of the ropes (in yards);

$$\text{then } 4y = 2x + 280,$$

$$\text{and } \frac{5y}{5x} = \frac{y}{3x} + 12;$$

$$\text{or by transposition, } \frac{2y}{3x} = 12;$$

$$\therefore\ 4y = 72x,$$

$$\text{and } \therefore\ 72x = 2x + 280;$$

$$\text{by transposition, } 70x = 280,$$

$$\text{and } x = 4;$$

$$\therefore\ y = 72;$$

$\therefore$ the circumferences of the wheels are 20 and 12 yards, and the length of the strings 360 and 72 yards.

25. Three guineas were to be raised on two estates, to be charged proportionably to their values. Of this sum, *A*'s estate, which was 4 acres more than *B*'s, but worse by 2 shillings an acre, paid £1. 15*s*. But had *A* possessed 6 acres more, and *B*'s land been worth 3 shillings an acre less, it would have paid £2. 5*s*. Required the values of the estates.

Let x = the number of acres *B* had;

then $x + 4$ = the number *A* had.

Let y = the value of an acre of *A*'s land;

$\therefore\ y + 2$ = the value of an acre of *B*'s,

$$\text{and } (x + 4)\,.\,y : x\,.\,(y + 2) :: 35 : 28 :: 5 : 4;$$

$$\therefore\ 4xy + 16y = 5xy + 10x,$$

$$\text{and } xy = 16y - 10x.$$

$$\text{Again, } (x + 10)\,.\,y : x\,.\,(y - 1) :: 45 : 18 :: 5 : 2;$$

$$\therefore\ 2xy + 20y = 5xy - 5x,$$

$$\text{and by transposition, } 3xy = 20y + 5x,$$

$$\text{whence } 48y - 30x = 20y + 5x;$$

by transposition, $28y = 35x$,

and $4y = 5x$,

which value substituted in the first equation, gives

$$xy = 20x - 10x = 10x;$$

$$\text{and } \therefore y = 10,$$

$$\text{whence } x = \frac{4y}{5} = 8;$$

$\therefore$ the value of *A*'s estate $= (x + 4) \,.\, y = 120 =$ £6; and the value of *B*'s $= x \,.\, (y + 2) = 96 =$ £4. 16*s*.

26. A coach set out from Cambridge to London with a certain number of passengers, 4 more being on the outside than within. Seven outside passengers could travel at 2 shillings less expense than 4 inside. The fare of the whole amounted to £9. But at the end of half the journey, it took up 3 more outside and one more inside passengers; in consequence of which the fare of the whole became increased in the proportion of 17 to 15. Required the number of passengers, and the fare of the inside and outside.

$$\left.\begin{array}{l}\text{Let } x = \text{the number of inside} \\ \therefore x + 4 = \text{the number of outside}\end{array}\right\} \text{passengers,}$$

and $y =$ the fare of an outside passenger;

$$\therefore \frac{7y + 2}{4} = \text{the fare of an inside passenger,}$$

$$\text{and } \frac{7xy + 2x}{4} + y \,.\, (x + 4) = 180.$$

Also $\frac{3y}{2} + \frac{7y + 2}{8} =$ the fare of the passengers taken up half-way $= \frac{19y + 2}{8}$;

$$\therefore \frac{19y + 2}{8} : 180 :: 2 : 15,$$

$$\text{or } \frac{19y + 2}{8} : 12 :: 2 : 1;$$

$$\therefore \frac{19y + 2}{8} = 24,$$

$$\text{and } 19y + 2 = 192;$$

$\therefore$ by transposition, $19y = 190$,
and $y = 10$;

$\therefore$ from the first equation, $\dfrac{70x + 2x}{4} + 10 \,.\, (x + 4) = 180$;

by transposition, $28x = 140$,
and $x = 5$;

$\therefore$ there were 5 inside, and 9 outside passengers, and the fares were 18 and 10 shillings, respectively.

27. In one of the corners of a rectangular garden there is a fish-pond, whose area is one-ninth part of the whole garden; the periphery of the garden exceeding that of the fish-pond by 200 yards. Also if the greater side be increased by 3 yards, and the other by 5 yards, the garden will be enlarged by 645 square yards. The fish-pond is a rectangle about the same diameter with the garden. Required the periphery of the garden, and the length of each side.

Let $x =$ the length of the lesser side,
and $y =$ the length of the greater;

$\therefore \dfrac{x}{3}$ and $\dfrac{y}{3} =$ the lengths of the lesser and greater sides of the fish-pond, (*Eucl.* B. VI. Prop. 24.)

Also $2 \,.\, (x + y) =$ the periphery of the garden;

and $\dfrac{2 \,.\, (x + y)}{3} =$ the periphery of the fish-pond;

$$\therefore\ 2 \,.\, (x + y) - \frac{2}{3} \,.\, (x + y) = 200,$$

$$\text{or } \frac{2}{3} \,.\, (x + y) = 100;$$

$$\therefore\ x + y = 150.$$

Also $(y + 3) \,.\, (x + 5) = xy + 645$;

$\therefore$ by transposition, $3x + 5y = 630$,

but from the former equation, $3x + 3y = 450$;

$\therefore$ by subtraction, $2y = 180$,
and $y = 90$;

$\therefore x = 150 - y = 60$;

and $\therefore$ the periphery $= 300$ yards, and the sides are 60 and 90 yards.

28. A and B each bought £300 into the stocks, A into the *three per cents.*, and B into the *fours.* These stocks were at such a price that B received one pound interest more than A. When afterwards each of the stocks rose 10 *per cent.*, they sold out their money, and A found himself £10 richer than B. Required the prices of the stocks.

Let $x =$ the price of the three per cents.,
and $y =$ the price of the fours;

$$\therefore\ x : 300 :: 3 : A\text{'s interest} = \frac{900}{x},$$

$$\text{and } y : 300 :: 4 : B\text{'s interest} = \frac{1200}{y};$$

$$\therefore\ \frac{900}{x} + 1 = \frac{1200}{y}.$$

Again, $x : x + 10 :: 300 :$ what A received when selling $= \dfrac{300 \,.\, (x + 10)}{x}$;

and in the same way B received $\dfrac{300 \,.\, (y + 10)}{y}$;

$$\therefore\ \frac{300 \,.\, (x + 10)}{x} = \frac{300 \,.\, (y + 10)}{y} + 10,$$

$$\text{and } 30 + \frac{300}{x} = 30 + \frac{300}{y} + 1;$$

$$\therefore\ \frac{300}{x} = \frac{300}{y} + 1,$$

and $\dfrac{900}{y} + 4 = \left(\dfrac{900}{x} + 1 = \right)\dfrac{1200}{y}$ from the first equation;

$$\therefore \text{ by transposition, } 4 = \frac{300}{y},$$

$$\text{and } y = 75;$$

$$\therefore\ \frac{300}{x} = 5,$$

and consequently, $x = 60$;

$\therefore$ the prices of the stocks were 60 and 75 *per cent.*

29. £500 was to be lent out at simple interest in two separate sums, the smaller at 2 *per cent.* more than the other. The interest of the greater sum was afterwards increased, and

that of the smaller sum diminished by 1 *per cent.* By this, the interest of the whole was augmented by one-fourth of the former value. But if the interest of the greater sum had been so increased, without any diminution of the less, the interest of the whole would have been increased one-third. What were the sums and the rate *per cent.* of each?

Let $x =$ the less sum;

$\therefore\ 500 - x =$ the greater;

let $y + 1 =$ the interest of the less;

$\therefore\ y - 1 =$ the interest of the greater,

and $\frac{x \cdot (y+1)}{100} + \frac{(500 - x) \cdot (y - 1)}{100} = \frac{x}{50} + 5y - 5 =$ the former interest,

and $\frac{xy}{100} + \frac{(500 - x) \cdot y}{100} = 5y =$ the second interest;

$$\therefore\ 5y = \frac{5}{4} \cdot \left(\frac{x}{50} + 5y - 5,\right)$$

$$\text{and } \therefore\ 4y = \frac{x}{50} + 5y - 5;$$

$$\text{by transposition, } y = 5 - \frac{x}{50}.$$

Again, the third interest $= \frac{x \cdot (y+1)}{100} + \frac{(500 - x) \cdot y}{100} = \frac{x}{100} + 5y$;

$$\therefore\ \left(\frac{x}{100} + 5y\right) \cdot 3 = 4 \cdot \left(\frac{x}{50} + 5y - 5\right),$$

$$\text{or } \frac{3x}{100} + 15y = \frac{8x}{100} + 20y - 20;$$

by transposition, $\left(20 - \frac{5x}{100} =\right) 20 - \frac{x}{20} = 5y = 25 - \frac{x}{10}$ from the former equation;

$$\therefore\ \frac{x}{20} = 5,$$

$$\text{and } x = 100;$$

$$\text{also } y + 1 = 6 - \frac{x}{50} = 4;$$

$\therefore$ the sums were 100 and 400 pounds, and the rates of interest £4 and £2, respectively.

SECTION VIII.

Examples of the Solution of Problems producing pure Equations.

1. WHAT two numbers are those, whose sum is to the greater as 10 to 7; and whose sum multiplied by the less produces 270?

$$\begin{aligned} \text{Let } 10x &= \text{their sum;} \\ \therefore\ 7x &= \text{the greater number,} \\ \text{and } 3x &= \text{the less;} \\ \text{whence } 30x^2 &= 270, \\ \text{and } x^2 &= 9; \\ \therefore\ x &= \pm 3, \end{aligned}$$

and the numbers are ± 21 and ± 9.

2. There are two numbers in the proportion of 4 to 5, the difference of whose squares is 81. What are those numbers?

$$\begin{aligned} \text{Let } 4x \text{ and } 5x &= \text{the numbers;} \\ \text{then } (25x^2 - 16x^2 =)\ 9x^2 &= 81; \\ \therefore\ x^2 &= 9, \\ \text{and } x &= \pm 3, \end{aligned}$$

and the numbers are ± 12 and ± 15.

3. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

$$\begin{aligned} \text{Let } 2x &= \text{their difference;} \\ \therefore\ 9x &= \text{the greater,} \\ \text{and } 7x &= \text{the less;} \\ \therefore\ (81x^2 - 49x^2 =)\ 32x^2 &= 128, \\ \text{and } x^2 &= 4; \\ \therefore\ x &= \pm 2, \end{aligned}$$

and the numbers are ± 18 and ± 14.

4. A Mercer bought a piece of silk for £16. 4s.; and the number of shillings which he paid for a yard was to the number of yards as 4 : 9. How many yards did he buy, and what was the price of a yard?

Let $4x =$ the number of shillings he paid for a yard;

$\therefore\ 9x =$ the number of yards,

and $36x^2 =$ (the price of the whole $=$) 324;

$$\therefore\ x^2 = 9,$$

$$\text{and } \therefore\ x = \pm 3;$$

consequently there were 27 yards, at 12s. per yard.

5. It is required to divide the number 18 into two such parts, that the squares of those parts may be in the proportion of 25 to 16.

Let $x =$ the greater part;

then $18 - x =$ the less;

$$\therefore\ x^2 : (18 - x)^2 :: 25 : 16,$$

and (*Alg.* 179, 9.) $x : 18 - x :: 5 : 4;$

$$\therefore\ x : 18 :: 5 : 9,$$

$$\text{and } x : 2 :: 5 : 1;$$

$$\therefore\ x = 10,$$

and the parts are 10 and 8.

6. Find three numbers in the proportion of $\frac{1}{2}, \frac{2}{3}$, and $\frac{3}{4}$; the sum of whose squares is 724.

Reducing the fractions to a common denominator, the required numbers will evidently be in the proportion of 6, 8, and 9;

let $\therefore\ 6x, 8x$, and $9x$, represent the numbers;

then $(36x^2 + 64x^2 + 81x^2 =)\ 181x^2 = 724;$

$$\therefore\ x^2 = 4,$$

$$\text{and } x = \pm 2,$$

and consequently, the numbers are ± 12, ± 16, and ± 18.

7. It is required to divide the number 14 into two such parts, that the quotient of the greater part divided by the less,

may be to the quotient of the less divided by the greater as 16 : 9.

$$\text{Let } x = \text{the greater part;}$$
$$\therefore 14 - x = \text{the less,}$$
$$\text{and } \frac{x}{14-x} : \frac{14-x}{x} :: 16 : 9;$$
$$\text{or } x^2 : (14-x)^2 :: 16 : 9;$$
$$\therefore (\textit{Alg. } 179, 9.)\ x : 14 - x :: 4 : 3,$$
$$\text{and } x : 14 :: 4 : 7;$$
$$\therefore x : 2 :: 4 : 1,$$
$$\text{and } x = 8;$$
$$\therefore \text{the parts are 8 and 6.}$$

8. What two numbers are those, whose difference is to the less as 4 to 3; and their product multiplied by the less is equal to 504?

$$\text{Let } 4x = \text{the difference;}$$
$$\text{then } 3x = \text{the less,}$$
$$\text{and } 7x = \text{the greater;}$$
$$\text{whence } 63x^3 = 504,$$
$$\text{or } x^3 = 8;$$
$$\therefore x = 2;$$
$$\text{and the numbers are 14 and 6.}$$

9. What two numbers are as 5 to 4, the sum of whose cubes is 5103?

$$\text{Let } 5x \text{ and } 4x = \text{the numbers;}$$
$$\therefore (125x^3 + 64x^3 =)\ 189x^3 = 5103,$$
$$\text{and } x^3 = 27;$$
$$\therefore x = 3,$$
$$\text{and the numbers are 15 and 12.}$$

10. A number of boys set out to rob an orchard, each carrying as many bags as there were boys in all, and each bag capable of containing 4 times as many apples as there were boys. They filled their bags, and found the number of apples was 2916. How many boys were there?

$$\begin{aligned} \text{Let } x &= \text{the number of boys;} \\ \text{then } x^2 &= \text{the number of bags,} \\ \text{and } 4x^3 &= \text{the number of apples;} \\ \therefore\ 4x^3 &= 2916, \\ \text{and } x^3 &= 729; \\ \therefore\ x &= 9; \end{aligned}$$

$\therefore$ there were 9 boys.

11. A person bought for one crown as many pounds of sugar as were equal to half the number of crowns he laid out. In selling the sugar he received for every 25 lbs. as many crowns as the whole had cost him; and he received on the whole 20 crowns. How many crowns did he lay out, and what did he give for a pound?

$$\begin{aligned} \text{Let } 2x &= \text{the number of crowns he laid out;} \\ \therefore\ x &= \text{the number of lbs. for one crown.} \\ \text{and } 2x^2 &= \text{the number of lbs. in all,} \\ \text{and } \frac{2x}{25} &= \text{the selling price of one lb.;} \end{aligned}$$

$$\therefore\ 2x^2 \times \frac{2x}{25} = 20,$$

$$\text{and } x^3 = 125;$$

$$\text{whence } x = 5;$$

$\therefore$ he laid out 10 crowns, and gave one shilling for a lb.

12. A detachment from an army was marching in regular column with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. Required the number of men.

$$\begin{aligned} \text{Let } x &= \text{the number in front;} \\ \therefore\ x + 5 &= \text{the number in depth,} \\ \text{and } x^2 + 5x &= 5x + 4225, \\ \therefore\ x^2 &= 4225, \\ \text{and } x &= \pm 65; \end{aligned}$$

$\therefore$ the number of men $= 5x + 4225 = 4550$, the negative value not answering the conditions of the problem.

13. A number of shillings were placed at equal distances on a table, so as to form the sides of an equilateral triangle; then from the middle of each side a number of shillings, equal to the square root of the number in the side, were taken, and placed upon the corner shilling opposite to that side; it then appeared that the number on each side was to the number previously upon it, as 5 to 4. Required the number of shillings on one side at first.

Let $x^2 =$ the number;

then $x^2 + x : x^2 :: 5 : 4$,

and $x : x^2 :: 1 : 4$;

$\therefore x^2 = 4x$,

and $x = 4$;

whence $x^2 = 16 =$ the number required.

14. A certain sum of money is divided every week among the resident members of a corporation. It happened one week that the number resident was the square root of the number of pounds to be divided. Two men, however, coming into residence the week after, diminished the dividend of each of the former individuals £1. 6*s*. 8*d*. What was the sum to be divided?

Let $x^2 =$ the number of pounds;

then $x =$ the number of men resident, and also $=$ the sum each received.

$$\text{Hence } x - \frac{4}{3} = \frac{x^2}{x+2};$$

$$\text{or } x^2 + \frac{2}{3}x - \frac{8}{3} = x^2;$$

$$\text{by transposition, } \frac{2}{3}x = \frac{8}{3},$$

and $x = 4$;

$\therefore x^2 = 16 =$ the sum required.

15. Two partners A and B dividing their gain (£60), B took £20; A's money continued in trade 4 months, and if the number 50 be divided by A's money, the quotient will give

the number of months that *B*'s money, which was £100, continued in trade. What was *A*'s money, and how long did *B*'s money continue in trade?

Suppose *A*'s money was x pounds;

$\therefore \frac{50}{x}$ = the number of months *B*'s money was in trade,

and since *B* gained £20, *A* gained £40;

$$\therefore 4x : \frac{5000}{x} :: 2 : 1;$$

$$\text{or } x : \frac{2500}{x} :: 1 : 1;$$

$$\therefore x^2 = 2500,$$

$$\text{and } x = \pm 50;$$

$\therefore$ *A*'s money was £50, and *B*'s money was one month in trade.

16. Two workmen *A* and *B* were engaged to work for a certain number of days at different rates. At the end of the time, *A* who had played 4 of those days, had 75 shillings to receive; but *B* who had played 7 of those days, received only 48 shillings. Now had *B* only played 4 days, and *A* played 7 days, they would have received exactly alike. For how many days were they engaged; how many did each work, and what had each *per* day?

Let x = the number of days for which they were engaged;

$\therefore x - 4$ = the number *A* worked,

and $x - 7$ = the number *B* worked,

and $\frac{75}{x-4}$ = the number of shillings *A* received *per* day;

and $\frac{48}{x-7}$ = the number of shillings *B* received *per* day;

$$\therefore \frac{75 \cdot (x-7)}{x-4} = \frac{48 \cdot (x-4)}{x-7},$$

$$\text{and } 25 \cdot (x-7)^2 = 16 \cdot (x-4)^2;$$

$$\therefore 5 \cdot (x-7) = \pm 4 \cdot (x-4);$$

$$\therefore x = 19, \text{ or } \frac{17}{3};$$

and $\therefore$ they were engaged to work 19 days,
and A worked 15, and B 12 days,
and A received 5 shillings, and B 4 shillings *per* day.

17. Two Travellers A and B set out to meet each other, A leaving the town C at the same time that B left D. They travelled the direct road CD, and on meeting it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What was the distance between C and D?

Let $x =$ the number of miles A has travelled;
$\therefore\ x - 18 =$ the number B has travelled,
and $x - 18 : x :: 15\frac{3}{4}$: the number of days A travelled
$= \dfrac{63x}{4 \cdot (x - 18)}$;
also $x : x - 18 :: 28$: the number of days B travelled
$= \dfrac{28 \cdot (x - 18)}{x}$;

$$\therefore\ \frac{28 \cdot (x - 18)}{x} = \frac{63x}{4 \cdot (x - 18)};$$
$$\text{or } 16 \cdot (x - 18)^2 = 9x^2;$$
$$\therefore\ 4 \cdot (x - 18) = \pm 3x,$$
$$\text{and } x = 72, \text{ or } 10\tfrac{2}{7};$$

whence A travelled 72, and B 54 miles;
and $\therefore$ the whole distance CD 126 miles.

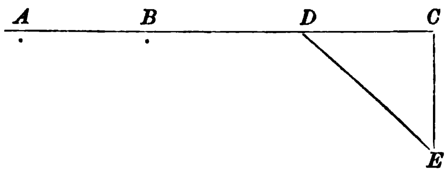
18. A and B lay out some money on speculation. A disposes of his bargain for £11, and gains as much *per* cent. as B lays out; B's gain is £36, and it appears that A gains four times as much *per* cent. as B. Required the capital of each.

Let $4x = B$'s capital, and $\therefore$ A's gain *per* cent.;
then $x = B$'s gain *per* cent.,
and $100 : 4x :: x : 36$;

$$\therefore\ 4x^2 = 36 \times 100,$$
$$\text{and } x^2 = 9 \times 100;$$
$$\therefore\ x = \pm 30,$$
$$\text{and } \therefore\ B\text{'s capital} = 120,$$
$$\text{and } 220 : 100 :: 11 : A\text{'s capital} = \frac{11 \times 10}{22} = 5.$$

19. **The Captain of a privateer descrying a trading vessel 7 miles ahead, sailed 20 miles in direct pursuit of her, and then observing the trader steering in a direction perpendicular to her former course, changed his own course so as to overtake her without making another tack. On comparing their reckonings, it was found that the privateer had run at the rate of 10 knots in an hour, and the trading vessel at the rate of 8 knots in the same time. Required the distance sailed by the privateer.**

Let A, B, be the original places of the privateer and trader, E the point of concourse, D the place where the



captain changed his course, CE being perpendicular to AC. $AB = 7$, $AD = 20$.

Now $(10 : 8 ::)\ 5 : 4 ::$ the velocity of the privateer : the velocity of the trader

$$:: AD : BC :: 20 : BC;$$
$$\therefore\ BC = \frac{20 \times 4}{5} = 16;$$
$$\therefore\ DC = 16 - BD = 16 - 13 = 3,$$
$$\text{and } DE : CE :: 5 : 4. \quad \text{Let } CE = x;$$
$$\therefore\ \sqrt{(9 + x^2)} : x :: 5 : 4,$$
$$\text{and } 9 + x^2 : x^2 :: 25 : 16;$$
$$\therefore\ 9 : x^2 :: 9 : 16;$$
$$\therefore\ x^2 = 16, \text{ and } x = \pm 4,$$
$$\text{and } \therefore\ DE = 5, \text{ and } AD + DE = 25.$$

20. A Vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on, for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Let $x =$ the number of gallons drawn the first time;

$\therefore\ 256 - x =$ the quantity of wine left,

and $256 : 256 - x :: x :$ the quantity of wine drawn the second time $= \dfrac{x \cdot (256 - x)}{256}$;

$$\therefore\ 256 - x - \frac{x \cdot (256 - x)}{256} = \frac{(256 - x)^2}{256} = \text{the}$$

quantity left after the second draught.

In the same way, $\left(\dfrac{256 - x}{256}\right)^2 \cdot x =$ the quantity drawn the third time,

and $\dfrac{(256 - x)^3}{\overline{256}|^2} =$ the quantity left,

and $\left(\dfrac{256 - x}{256}\right)^3 \cdot x$ and $\dfrac{(256 - x)^4}{\overline{256}|^3} =$ the quantities drawn and left the fourth time;

$$\text{whence } \frac{(256 - x)^4}{\overline{256}|^3} = 81,$$

and $256 - x = \overline{256}|^{\frac{3}{4}} \times 3 = 64 \times 3 = 192$;

$\therefore$ by transposition, $64 = x$,

and the quantities drawn off each time were 64, 48, 36, and 27 gallons.

21. What two numbers are those, whose difference multiplied by the greater produces 40, and by the less 15?

Let $x =$ the greater,

and $y =$ the less;

$\therefore\ x^2 - xy = 40$,

and $xy - y^2 = 15$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 25$,
and $x - y = \pm 5$;
$\therefore$ from the first equation, $\pm 5x = 40$,
and $x = \pm 8$;
and from the second equation, $\pm 5y = 15$,
and $y = \pm 3$;
$\therefore$ the numbers are ± 8, and ± 3.

22. What two numbers are those, whose difference multiplied by the less, produces 42, and by their sum 133?

Let $x =$ the greater,
and $y =$ the less;
$\therefore (x - y) . y = 42$,
and $(x - y) . (x + y) = 133$;
$\therefore$ by subtraction, $(x - y) . x = 91$, subtracting the first equation from this, $(x - y) . (x - y) = 49$;
or $(x - y)^2 = 49$;
$\therefore x - y = \pm 7$;
whence $\pm 7y = 42$,
and $y = \pm 6$,
and $x = \pm 7 + y = \pm 13$,
$\therefore$ the numbers are ± 13, and ± 6.

23. In a mixture of rum and brandy, the difference between the quantities of each is to the quantity of brandy as 100 is to the number of gallons of rum; and the same difference is to the quantity of rum as 4 to the number of gallons of brandy. How many gallons are there of each?

Let $x =$ the number of gallons of rum,
and $y =$ the number of gallons of brandy;
$\therefore x - y : y :: 100 : x$,
and $x : x - y :: y : 4$;
$\therefore$ *ex æquali*, $x : y :: 25y : x$,
and $x^2 = 25y^2$;
$\therefore x = \pm 5y$, the negative value not answering the conditions of the problem.

Now from the second proportion $5y : 4y :: y : 4$;

$$\text{or } 5 : 1 :: y : 1;$$

$$\therefore y = 5,$$

$$\text{and } x = 25;$$

$\therefore$ there are 25 gallons of rum, and 5 of brandy.

24. What two numbers are those, whose difference being multiplied by the greater, and the product divided by the less, quotes 24; but if their difference be multiplied by the less, and the product divided by the greater, the quotient is 6?

Let $x =$ the greater,

and $y =$ the less;

$$\text{then } (x - y) \cdot \frac{x}{y} = 24,$$

$$\text{and } (x - y) \cdot \frac{y}{x} = 6;$$

dividing the first equation by the second, $\frac{x^2}{y^2} = 4$;

$$\therefore \frac{x}{y} = \pm 2;$$

$$\text{or } x = \pm 2y,$$

and $\therefore$ in the first case, $(x - y =)\, y = 12$, and $x = 24$;

but if $x = -2y$; $-3y \times -2 = 24$;

$$\therefore y = 4, \text{ and } x = -8.$$

25. It is required to find two numbers such, that the product of the greater and square root of the less may be equal to 48, and the product of the less and square root of the greater may be 36.

Let x^2 and y^2 be the two numbers;

$$\therefore x^2 y = 48, \text{ and } xy^2 = 36;$$

$$\therefore \frac{48}{x} = (xy =) \frac{36}{y},$$

$$\text{or } \frac{4}{x} = \frac{3}{y};$$

$$\therefore y = \frac{3x}{4};$$

$$\text{whence } \frac{3x^3}{4} = 48,$$

$$\text{and } x^3 = 64;$$

$$\therefore x = 4,$$

$$\text{and consequently, } y = 3;$$

$$\therefore \text{ the numbers are } 16, \text{ and } 9.$$

26. Find two numbers such, that the square of the greater multiplied by the less may be equal to 448, and the square of the less multiplied by the greater may be 392.

Let $x =$ the greater, and $y =$ the less;

then $x^2y = 448$, and $xy^2 = 392$;

$$\therefore \frac{448}{x} = (xy =) \frac{392}{y},$$

$$\text{or } \frac{8}{x} = \frac{7}{y};$$

$$\therefore x = \frac{8y}{7},$$

$$\text{and consequently, } \frac{8y^3}{7} = 392,$$

$$\text{or } \frac{y^3}{7} = 49;$$

$$\therefore y^3 = 343, \text{ and } y = 7;$$

$$\therefore x = 8;$$

$$\therefore \text{ the numbers are } 8, \text{ and } 7.$$

27. *A* and *B* carried 100 eggs between them to market, and each received the same sum. If *A* had carried as many as *B*, he would have received 18 pence for them, and if *B* had taken only as many as *A*, he would have received only 8 pence. How many had each?

Let $x =$ the number *A* had,

and $y =$ the number *B* had;

then $\frac{18}{y} =$ the price of one egg of *A*'s (in pence),

and $\frac{8}{x} =$ the price of one of *B*'s;

$$\therefore \frac{18x}{y} = \frac{8y}{x},$$
$$\text{and } 9x^2 = 4y^2;$$

$\therefore$ $3x = \pm 2y$, the negative value of which will not answer the conditions of the problem.

$$\text{Now } (x + y =)\, x + \frac{3x}{2} = 100;$$
$$\therefore (2x + 3x =)\, 5x = 200,$$
$$\text{and } x = 40;$$
$$\text{and } \therefore y = 60.$$

28. What two numbers are those, which being both multiplied by 27 the first product is a square, and the second the root of that square: but being both multiplied by 3, the first product is a cube, and the second the root of that cube?

Let x and y be the numbers;
$$\text{then } \sqrt{(27x)} = 27y,$$
$$\text{and } \therefore x = 27y^2;$$
$$\text{also } \sqrt[3]{(3x)} = 3y,$$
$$\text{and } \therefore x = 9y^3;$$
$$\text{whence } 9y^3 = 27y^2,$$
$$\text{and } y = 3;$$
$$\therefore x = 27y^2 = 243;$$
$\therefore$ the numbers are 243, and 3.

29. It is required to find the three sides of a right-angled triangle from the following data. The number of square feet in the area is equal to the number of feet in the hypothenuse + the sum in the other two sides; and the square described upon the hypothenuse is less than the square described upon a line equal in length to the two sides, by half the product of the numbers representing the base and area.

Let x = the number of feet in the altitude,
and y = the number in the base;

$\therefore \sqrt{(x^2 + y^2)}$ = the number in the hypothenuse, (*Eucl.* B. I. p. 47.)

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$$\text{and } \frac{xy}{2} = \text{the area};$$

$$\therefore \tfrac{1}{2}xy = \sqrt{(x^2 + y^2)} + x + y;$$

$$\text{also } x^2 + y^2 = \{(x + y)^2 - \tfrac{1}{4}xy^2 =\}\ x^2 + 2xy + y^2 - \tfrac{1}{4}xy^2;$$

$$\therefore \text{by transposition, } \tfrac{1}{4}xy^2 = 2xy,$$

$$\text{and } y = 8;$$

$$\text{hence from the first equation, } 4x = \sqrt{(x^2 + 64)} + x + 8,$$

$$\text{and by transposition, } 3x - 8 = \sqrt{(x^2 + 64)};$$

$$\therefore 9x^2 - 48x + 64 = x^2 + 64,$$

$$\text{and } 8x^2 = 48x;$$

$$\therefore x = 6;$$

$$\text{whence the hypothenuse} = \sqrt{(64 + 36)} = 10;$$

$\therefore$ the sides are 6, 8, and 10 feet, respectively.

30. A Farmer has 2 cubical stacks of hay. The side of one is 3 yards longer than the side of the other; and the difference of their contents is 117 solid yards. Required the side of each.

Let x = the side of the greater,
and y = the side of the less;

$$\therefore x^3 - y^3 = 117,$$

$$\text{and } x - y = 3;$$

$$\text{cubing the latter equation, } x^3 - 3x^2y + 3xy^2 - y^3 = 27;$$

$$\text{but } x^3 \qquad\qquad\qquad - y^3 = 117;$$

$$\therefore \text{by subtraction, } 3x^2y - 3xy^2 = 90,$$

$$\text{and } xy \,.\, (x - y) = 30,$$

$$\text{or } 3xy = 30;$$

$$\therefore xy = 10.$$

$$\text{Now } x^2 - 2xy + y^2 = 9,$$

$$\text{and } 4xy \qquad = 40;$$

$$\therefore \text{by addition, } x^2 + 2xy + y^2 = 49,$$

$$\text{and } x + y = \pm 7;$$

$$\text{but } x - y = 3;$$

$$\therefore \text{by addition, } 2x = 10, \text{ or } -4;$$

$$\therefore x = 5, \text{ or } -2,$$

$$\text{and by subtraction, } 2y = 4, \text{ or } -10;$$

$$\therefore y = 2, \text{ or } -5,$$

and the sides of the stacks are 5, and 2 yards, respectively.

31. When a parish was enclosed, the allotment of one of the proprietors consisted of two pieces of ground; one of which was in the form of a right-angled triangle; the other was a rectangle, one of the sides of which was equal to the hypothenuse of the triangle, the other, to half the greater side; but wishing to have his land in one piece, he exchanged his allotments for a square piece of ground of equal area, one side of which equalled the greater of the sides of the triangle which contained the right angle. By this exchange he found that he had saved ten poles of railing. What are the respective areas of the triangle and rectangle; and what is the length of each of their sides?

Let $2x =$ the greater side of the triangle,

and $y =$ the less;

$\therefore \sqrt{(4x^2 + y^2)} =$ the hypothenuse; and also the greater side of the rectangle,

and $x =$ the less side of the rectangle;

$\therefore xy =$ the area of the triangle,

and $x\sqrt{(4x^2 + y^2)} =$ the area of the rectangle;

$$\therefore 4x^2 = xy + x\sqrt{(4x^2 + y^2)},$$

$$\text{or } 4x - y = \sqrt{(4x^2 + y^2)};$$

also $8x + 10 = 2x + y + \sqrt{(4x^2 + y^2)} + 2x + 2\sqrt{(4x^2 + y^2)}$, or $4x + 10 = y + 3\sqrt{(4x^2 + y^2)}$; in which equation substituting the value of $\sqrt{(4x^2 + y^2)}$ found above;

$$\therefore 4x + 10 = y + 3(4x - y) = 12x - 2y;$$

$\therefore$ by transposition, $2y = 8x - 10$,

and $y = 4x - 5$;

$\therefore$ from the first equation, $5 = \sqrt{\{4x^2 + (4x - 5)^2\}}$,

and $25 = 4x^2 + 16x^2 - 40x + 25$;

by transposition, $40x = 20x^2$;

$\therefore 2 = x$,

and $y = 4x - 5 = 3$;

$\therefore$ the sides of the triangle are 3, 4, and 5; the sides of the rectangle are 2, and 5; and the areas of the triangle and rectangle are 6, and 10, respectively.

SECTION IX.

Examples of the Solution of Problems producing Adfected Quadratic Equations.

1. A MERCHANT sold a quantity of brandy for £39, and gained as much *per* cent. as the brandy cost him. What was the price of the brandy?

$$\text{Let } x = \text{the price of the brandy;}$$

$$\text{then } 100 : x :: x : \text{the gain} = \frac{x^2}{100},$$

$$\text{and } \therefore \frac{x^2}{100} = 39 - x,$$

$$\text{or } x^2 = 3900 - 100x;$$

$$\text{by transposition, } x^2 + 100x = 3900,$$

$$\text{completing the square, } x^2 + 100x + (50)^2 = 3900 + 2500$$
$$= 6400;$$

$$\text{extracting the root, } x + 50 = \pm 80;$$

$$\therefore x = 30, \text{ or } -130;$$

$$\therefore \text{the price was £30.}$$

2. There are two numbers whose difference is 9, and their sum multiplied by the greater produces 266. What are those numbers?

$$\text{Let } x = \text{the greater;}$$

$$\therefore x - 9 = \text{the less,}$$

$$\text{and } x \,.\, (2x - 9) = 266;$$

$$\therefore x^2 - \frac{9}{2} \,.\, x = \frac{266}{2};$$

$$\text{completing the square, } x^2 - \frac{9}{2}x + \frac{81}{16} = \frac{266}{2} + \frac{81}{16} = \frac{2209}{16};$$

$$\text{extracting the root, } x - \frac{9}{4} = \pm \frac{47}{4};$$

$$\therefore\ x = 14, \text{ or } -\frac{19}{2};$$

$$\therefore\ x - 9 = 5, \text{ or } -\frac{37}{2},$$

and both values answer the conditions of the problem.

3. It is required to find two numbers, the first of which may be to the second as the second is to 16; and the sum of the squares of the numbers may be equal to 225.

Let $x =$ the first number;

$\therefore\ \sqrt{(16x)} =$ the second,

and $x^2 + 16x = 225$;

completing the square, $x^2 + 16x + 64 = 225 + 64 = 289$;

extracting the root, $x + 8 = \pm 17$;

$\therefore\ x = 9, \text{ or } - 25$;

but as this latter value of x makes the second number an impossible quantity, 9 is the only value of x which answers the conditions, and therefore the numbers are 9 and 12.

4. Bought two sorts of linen for 6 crowns. An ell of the finer cost as many shillings as there were ells of the finer. Also 28 ells of the coarser (which was the whole quantity) were at such a price that 8 ells cost as many shillings as 1 ell of the finer. How many ells were there of the finer, and what was the value of each piece?

Let $x =$ the number of ells of the finer;

$\therefore\ x^2 =$ the price of the finer (in shillings),

and $8 : 28 :: x :$ the price of the coarser $= \frac{7x}{2}$;

$$\therefore\ x^2 + \frac{7x}{2} = 30;$$

completing the square, $x^2 + \frac{7}{2}x + \frac{49}{16} = 30 + \frac{49}{16} = \frac{529}{16}$;

extracting the root, $x + \frac{7}{4} = \pm\frac{23}{4}$,

and $x = 4, \text{ or } -\frac{15}{2}$,

and $\therefore$ the price of the finer $= 16$ shillings, and of the coarser $=$ 14 shillings.

5. Two partners A and B gained £18 by trade. A's money was in trade 12 months, and he received for his principal and gain £26. Also B's money, which was £30, was in trade 16 months. What money did A put into trade?

Let $x =$ the number of pounds he put in;

$\therefore 26 - x =$ the number he gained,

and $12x + 16 \times 30 : 12x :: 18 : 26 - x$;

$\therefore x + 40 : x :: 18 : 26 - x$,

and $18x = 1040 - 14x - x^2$;

by transposition, $x^2 + 32x = 1040$;

completing the square, $x^2 + 32x + (16)^2 = 1040 + 256$
$= 1296$;

extracting the root, $x + 16 = \pm 36$;

$\therefore x = 20$, or -52,

and consequently A put £20 into trade.

6. A person bought some sheep for £72; and found that if he had bought 6 more for the same money, he would have paid £1 less for each. How many did he buy, and what was the price of each?

Let $x =$ the number of sheep bought;

then $\frac{72}{x} =$ the price of one (in pounds),

and $\frac{72}{x+6} =$ the price of one, if he had bought 6 more;

$$\therefore \frac{72}{x+6} + 1 = \frac{72}{x};$$

$$\therefore 72x + x^2 + 6x = 72x + 432;$$

$$\therefore x^2 + 6x = 432;$$

completing the square, $x^2 + 6x + 9 = 441$;

extracting the root, $x + 3 = \pm 21$,

and $x = 18$, or -24,

and $\therefore$ he bought 18, and the price of one $= \frac{72}{18}$
$= 4$ pounds.

7. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame.

The area of the glass $= 12 \times 18 = 216$.

Let $x =$ the width of the frame (in inches);

then the area of the frame $= (18 + 2x) \,.\, (12 + 2x) - 216$,

$$\text{and } \therefore\ (18 + 2x) \,.\, (12 + 2x) - 216 = 216,$$

$$\text{or } 4x^2 + 60x = 216,$$

$$\text{and } x^2 + 15x = 54;$$

$$\text{completing the square, } x^2 + 15x + \left(\frac{15}{2}\right)^2 = 54 + \frac{225}{4} = \frac{441}{4};$$

$$\text{extracting the root, } x + \frac{15}{2} = \pm\frac{21}{2};$$

$$\therefore\ x = 3, \text{ or } -18,$$

and $\therefore$ the width must be 3 inches.

8. There are two square buildings, that are paved with stones, a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately?

Let x and $x + 12 =$ the number of feet in the sides of each;

$\therefore\ x^2$ and $(x + 12)^2 =$ the number of stones in the squares,

$$\text{and } x^2 + x^2 + 24x + 144 = 2120;$$

$$\text{by transposition, } 2x^2 + 24x = 1976,$$

$$\text{or } x^2 + 12x = 988;$$

$$\text{completing the square, } x^2 + 12x + 36 = 988 + 36 = 1024;$$

$$\text{extracting the root, } x + 6 = \pm 32;$$

$$\therefore\ x = 26, \text{ or } -38,$$

whence the lengths are 26, and 38 feet, respectively.

9. A labourer dug two trenches, one of which was 6 yards longer than the other, for £17. 16*s*. and the digging of each of them cost as many shillings *per* yard as there were yards in its length. What was the length of each?

Let x and $x + 6 =$ the number of yards in each;
$\therefore\ x^2 + (x + 6)^2 = 356$ shillings,
or $2x^2 + 12x + 36 = 356$;
by transposition, $2x^2 + 12x = 320$;
or $x^2 + 6x = 160$;
completing the square, $x^2 + 6x + 9 = 169$;
extracting the root, $x + 3 = \pm\ 13$,
and $x = 10$, or $-\ 16$;
$\therefore$ the lengths were 10, and 16 yards.

10. A company at a tavern had £8. 15*s.* to pay; but before the bill was paid, two of them sneaked off, when those who remained had each 10 shillings more to pay. How many were in the company at first?

Let $x =$ the number;

then $\dfrac{175}{x} =$ the number of shillings each had to pay at first,

and $\dfrac{175}{x-2} =$ the number each had to pay, after two had sneaked off;

$$\therefore\ 10 = \frac{175}{x-2} - \frac{175}{x} = 175 \times \left(\frac{1}{x-2} - \frac{1}{x}\right)$$

and $\therefore\ 10 . x . (x - 2) = 175 \times 2$,
or $x^2 - 2x = 35$;
completing the square, $x^2 - 2x + 1 = 36$;
extracting the root, $x - 1 = \pm\ 6$,
and $\therefore\ x = 7$, or $-\ 5$;
consequently there were 7 at first.

11. A grazier bought as many sheep as cost him £60; out of which he reserved 15, and sold the remainder for £54, gaining 2 shillings a head by them. How many sheep did he buy, and what was the price of each?

Let $x =$ the number;

$\therefore\ \dfrac{60}{x} =$ the price of each in pounds,

and $(x - 15) . \left(\dfrac{60}{x} + \dfrac{1}{10}\right) = 54$;

$$\text{or } (x - 15) \,.\, (600 + x) = 540x,$$
$$\text{and } x^2 + 585x - 9000 = 540x;$$
$$\text{by transposition, } x^2 + 45x = 9000;$$
$$\text{completing the square, } x^2 + 45x + \overline{\frac{45}{2}}\Big|^2 = 9000 + \frac{2025}{4}$$
$$= \frac{38025}{4};$$
$$\text{extracting the root, } x + \frac{45}{2} = \pm \frac{195}{2},$$
$$\text{and } x = 75, \text{ or } -120;$$
$$\text{and} \therefore \text{ the number bought was } 75,$$
$$\text{and the price} = \frac{4}{5}£ = 16 \text{ shillings.}$$

12. *A* and *B* set out from two towns which were at the distance of 247 miles, and travelled the direct road till they met. *A* went 9 miles a day; and the number of days, at the end of which they met, was greater by 3 than the number of miles which *B* went in a day. How many miles did each go?

Let x = the number of days they travelled;
$\therefore 9x$ = the number of miles *A* went,
and $247 - 9x$ = the number *B* went,
and $\frac{247 - 9x}{x}$ = the number *B* went *per* day;

$$\therefore x - 3 = \frac{247 - 9x}{x},$$
$$\text{and } x^2 - 3x = 247 - 9x;$$
$$\text{by transposition, } x^2 + 6x = 247;$$
$$\text{completing the square, } x^2 + 6x + 9 = 256,$$
$$\text{extracting the root, } x + 3 = \pm 16,$$
$$\text{and } x = 13, \text{ or } -19,$$
$$\text{and} \therefore A \text{ went } 117, \text{ and } B\ 130 \text{ miles.}$$

13. A person bought two pieces of cloth of different sorts; whereof the finer cost 4 shillings a yard more than the other; for the finer he paid £18; but the coarser, which exceeded the finer in length by 2 yards, cost only £16.

How many yards were there in each piece, and what was the price of a yard of each?

Let $x =$ the number of yards of the finer;

$\therefore x + 2 =$ the number of yards of the coarser,

and $\frac{18}{x} =$ the price of a yard of the finer (in pounds);

also $\frac{16}{x+2} =$ the price of a yard of the coarser;

$$\therefore \frac{18}{x} = \frac{16}{x+2} + \frac{1}{5},$$

and $90x + 180 = 80x + x^2 + 2x$;

by transposition, $x^2 - 8x = 180$;

completing the square, $x^2 - 8x + 16 = 180 + 16 = 196$;

extracting the root, $x - 4 = \pm 14$;

$$\therefore x = 18, \text{ or } - 10,$$

consequently, there were 18 yards of the finer, and 20 of the coarser; and the prices were £1, and 16 shillings, respectively.

14. *A* set out from *C* towards *D*, and travelled 7 miles a day. After he had gone 32 miles, *B* set out from *D* towards *C*, and went every day $\frac{1}{19}$th of the whole journey; and after he had travelled as many days as he went miles in one day, he met *A*. Required the distance of the places *C* and *D*.

Suppose the distance was x miles;

$\therefore \frac{x}{19} =$ the number of miles *B* travelled *per* day; and also $=$ the number of days he travelled before he met *A*.

$$\therefore \frac{x^2}{361} + 32 + \frac{7x}{19} = x;$$

by transposition, $\frac{x^2}{361} - \frac{12x}{19} = -32$;

completing the square, $\frac{x^2}{361} - \frac{12x}{19} + 36 = 36 - 32 = 4$;

extracting the root, $\frac{x}{19} - 6 = \pm 2$;

$$\therefore \frac{x}{19} = 8, \text{ or } 4,$$

and $x = 152$, or 76, both which values answer the conditions of the problem. The distance therefore of C from D was 152, or 76 miles.

15. A and B sold 130 ells of silk, (of which 40 ells were A's, and 90 B's,) for 42 crowns. Now A sold for a crown one-third of an ell more than B did. How many ells did each sell for a crown?

$$\left.\begin{array}{r}\text{Let } x = \text{the number } B \text{ sold}\\ \therefore\ x + \frac{1}{3} = \text{the number } A \text{ sold}\end{array}\right\}\text{for a crown,}$$

$$\text{and } x : 90 :: 1 : \text{the price of 90 ells} = \frac{90}{x},$$

$$\text{and } x + \frac{1}{3} : 40 :: 1 : \text{the price of 40 ells} = \frac{120}{3x+1};$$

$$\therefore\ 42 = \frac{90}{x} + \frac{120}{3x+1},$$

$$\text{or } 7 = \frac{15}{x} + \frac{20}{3x+1};$$

$$\text{whence, } 21x^2 + 7x = 45x + 15 + 20x;$$

$$\text{by transposition, } 21x^2 - 58x = 15,$$

$$\text{and } x^2 - \frac{58}{21}x = \frac{15}{21};$$

$$\text{completing the square, } x^2 - \frac{58}{21}.x + \left(\frac{29}{21}\right)^2 = \frac{15}{21} + \frac{841}{(21)^2} = \frac{1156}{(21)^2};$$

$$\text{extracting the root, } x - \frac{29}{21} = \pm\frac{34}{21};$$

$$\therefore\ x = 3, \text{ or } -\frac{5}{21};$$

whence, B sold 3 ells, and A $3\frac{1}{3}$, for a crown.

16. Three Merchants, A, B, and C, made a joint stock, by which they gained a sum less than that stock by £80. A's share of the gain was £60; and his contribution to the stock was £17 more than B's. Also B and C together contributed £325. How much did each contribute?

Let $x =$ the number of pounds that A contributed;

$\therefore\ x - 17 =$ the number that B contributed,

and $325 - (x - 17) = 342 - x =$ the number that C contributed;

$\therefore\ 325 + x =$ the whole stock,

and $325 + x - 80 = 245 + x =$ the whole gain;

$$\therefore\ 325 + x : x :: 245 + x : 60,$$

$$\text{and } x^2 + 245x = 60x + 19500;$$

$$\text{by transposition, } x^2 + 185x = 19500;$$

$$\text{completing the square, } x^2 + 185x + \left(\frac{185}{2}\right)^2 = 19500 + \frac{34225}{4} = \frac{112225}{4};$$

$$\text{extracting the root, } x + \frac{185}{2} = \pm\frac{335}{2},$$

$$\text{and } x = 75, \text{ or } -260;$$

$\therefore$ the stocks of A, B, and C were 75, 58, and 267 pounds, respectively.

17. The joint stock of 2 partners A and B was £416. A's money was in trade 9 months, and B's 6 months: when they shared stock and gain, A received £228, and B £252. What was each man's stock?

$$\text{Let } x = A\text{'s stock};$$

$$\therefore\ 228 - x = \text{his gain};$$

$$\text{also } 416 - x = B\text{'s stock};$$

$$\text{and } x - 164 = \text{his gain};$$

$$\text{and } \therefore\ 64 = \text{the whole gain},$$

$$\text{and } 9x + 6\,.\,(416 - x) : 9x :: 64 : 228 - x;$$

$$\text{or } 3x + 2\,.\,(416 - x) : 3x :: 64 : 228 - x;$$

$$\therefore\ 192x = (x + 832)\,.\,(228 - x) = 189696 - 604x - x^2;$$

$$\text{by transposition, } x^2 + 796x = 189696;$$

$$\text{completing the square, } x^2 + 796x + (398)^2 = 189696 + 158404 = 348100;$$

$$\text{extracting the root, } x + 398 = \pm 590;$$

$$\text{and } x = 192, \text{ or } -988;$$

$\therefore$ the stocks were £192, and £224.

18. A body of men were formed into a hollow square, three deep, when it was observed, that with the addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square.

Let x = the number of men in a side of the hollow square;

$$\therefore\ x^2 - (x - 6)^2 = \text{the whole number of men,}$$
$$\text{and } x^2 - (x - 6)^2 + 25 = (x^{\frac{1}{2}} + 22)^2,$$
$$\text{or } 12x - 36 + 25 = x + 44x^{\frac{1}{2}} + 484;$$
$$\therefore \text{ by transposition, } 11x - 44x^{\frac{1}{2}} = 495,$$
$$\text{or } x - 4x^{\frac{1}{2}} = 45;$$
$$\text{completing the square, } x - 4x^{\frac{1}{2}} + 4 = 49;$$
$$\text{extracting the root, } x^{\frac{1}{2}} - 2 = \pm 7;$$
$$\therefore\ x^{\frac{1}{2}} = 9, \text{ or } -5,$$
$$\text{and } x = 81, \text{ or } 25,$$
$$\text{and } \therefore \text{ the whole number} = 936.$$

19. A Mercer bought a number of pieces of two different kinds of silk for £92. 3*s*. There were as many pieces bought of each kind, and as many shillings paid *per* yard for them, as a piece of that kind contained yards. Now 2 pieces, one of each kind, together measured 19 yards. How many yards were there in each?

Let x = the number of yards in one piece; and $\therefore$ = the number of pieces, and also the number of shillings *per* yard;

$$\therefore\ 19 - x = \text{the number in the other,}$$
$$\text{and } x^3, \text{ and } (19 - x)^3 = \text{the whole prices of each kind;}$$
$$\therefore\ x^3 + (19 - x)^3 = 1843,$$
$$\text{or } 57x^2 - 1083x + 6859 = 1843;$$
$$\text{by transposition, } 57x^2 - 1083x = -5016;$$
$$\text{or } x^2 - 19x = -88;$$
$$\text{completing the square, } x^2 - 19x + \left(\frac{19}{2}\right)^2 = \frac{361}{4} - 88 = \frac{9}{4};$$

$$\text{extracting the root, } x - \frac{19}{2} = \pm\frac{3}{2};$$

$$\therefore\ x = 11, \text{ or } 8;$$

$$\therefore\ 19 - x = 8, \text{ or } 11,$$

both which values answer the conditions of the problem;

$\therefore$ there were 11 yards in one, and 8 in the other.

20. A square court-yard has a rectangular gravel-walk round it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk; and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 92. Required the area of the court.

Let x = the breadth of the walk (in yards),

$\therefore\ 6x - 2$ = the side of the court,

and $4x - 2$ = the side of the interior square;

$\therefore\ (6x - 2)^2 - (4x - 2)^2$ = the area of the walk,

and $20x^2 - 8x - 92 = 4 \times (6x - 2)$;

by transposition, $20x^2 - 32x = 84$;

$$\therefore\ x^2 - \frac{8}{5}x = \frac{21}{5};$$

$$\text{completing the square, } x^2 - \frac{8}{5}x + \frac{16}{25} = \frac{21}{5} + \frac{16}{25} = \frac{121}{25};$$

$$\text{extracting the root, } x - \frac{4}{5} = \pm\frac{11}{5};$$

$$\therefore\ x = 3, \text{ or } -\frac{7}{5},$$

and $(6x - 2)^2 = (16)^2 = 256$, the area required.

21. A Merchant bought 54 gallons of Cognac brandy, and a certain quantity of British. For the former he gave half as many shillings *per* gallon as there were gallons of British, and for the latter 4 shillings *per* gallon less. He sold the mixture at 10 shillings *per* gallon, and lost £28. 16*s*. by his bargain. Required the price of the Cognac, and the number of gallons of British.

Let $2x$ = the number of gallons of British;

$\therefore$ $x =$ the number of shillings one gallon of Cognac cost,

and $54x =$ the price of all the Cognac;

also $x - 4 =$ the number of shillings one gallon of British cost,

and $2x^2 - 8x =$ the price of all the British;

$$\therefore\ 2x^2 - 8x + 54x = 10 . (54 + 2x) + 576;$$

by transposition, $2x^2 + 26x = 1116$,

or $x^2 + 13x = 558$;

completing the square, $x^2 + 13x + \left(\frac{13}{2}\right)^2 = 558 + \frac{169}{4}$

$$= \frac{2041}{4};$$

extracting the root, $x + \frac{13}{2} = \pm \frac{49}{2}$,

and $x = 18$, or $- 31$;

$\therefore$ he bought 36 gallons of British: the Cognac cost 18 shillings *per* gallon,

and $\therefore$ the whole price $=$ £48. 12*s.*

22. During the time that the shadow on a sun-dial, which shows true time, moves from one o'clock to five, a clock, which is too fast a certain number of hours and minutes, strikes a number of strokes = that number of hours and minutes, and it is observed that the number of minutes is less by 41 than the square of the number which the clock strikes at the last time of striking. The clock does not strike twelve during the time. How much is it too fast?

Let $x =$ the number of hours too fast;

then the clock strikes $(x + 2) + (x + 3) + (x + 4) + (x + 5)$ times $= 4x + 14$,

and the number of minutes $= x^2 + 10x + 25 - 41 = x^2 + 10x - 16$;

$$\therefore\ x + x^2 + 10x - 16 = 4x + 14;$$

by transposition, $x^2 + 7x = 30$;

completing the square, $x^2 + 7x + \left(\frac{7}{2}\right)^2 = 30 + \frac{49}{4} = \frac{169}{4}$;

extracting the root, $x + \frac{7}{2} = \pm\frac{13}{2}$;

$\therefore\ x = 3$, or -10,

and the number of minutes $= 23$;

$\therefore$ the clock is too fast 3 hours and 23 minutes.

23. A Vintner sold 7 dozen of sherry and 12 dozen of claret for £50. He sold 3 dozen more of sherry for £10 than he did of claret for £6. Required the price of each.

Let x = the price of a dozen of sherry (in pounds);

$\therefore\ x : 10 :: 1 :$ the number of dozens of sherry for £10, $= \frac{10}{x}$,

and $\frac{10}{x} - 3 = \frac{10 - 3x}{x}$ = the number of dozens of claret for £6;

$\therefore\ \frac{10-3x}{x} : 1 :: 6 :$ the price of a dozen of claret $= \frac{6x}{10 - 3x}$;

$$\therefore\ 7x + \frac{72x}{10 - 3x} = 50,$$

and $70x - 21x^2 + 72x = 500 - 150x$;

by transposition, $292x - 21x^2 = 500$,

or $x^2 - \frac{292}{21}.x = -\frac{500}{21}$;

completing the square,

$$x^2 - \frac{292}{21}.x + \left(\frac{146}{21}\right)^2 = \frac{21316}{(21)^2} - \frac{500}{21} = \frac{10816}{(21)^2};$$

extracting the root, $x - \frac{146}{21} = \pm\frac{104}{21}$,

and $x = 2$, or $\frac{250}{21}$;

$\therefore$ the price of a dozen of sherry was £2, and the price of a dozen of claret $= \frac{6x}{10 - 3x} =$ £3.

24. *A* and *B* hired a pasture into which *A* put 4 horses, and *B* as many as cost him 18 shillings a week. Afterwards *B* put in two additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

Let $x =$ the number of *B*'s horses at first;

then $\frac{18}{x} =$ the pay of each *per* week (in shillings);

$\therefore \frac{72}{x} =$ what *A* paid,

and $\frac{72}{x} + 18 =$ the price of the pasture;

also $x + 6 =$ the whole number of horses in the second case;

$$\therefore x + 6 : x + 2 :: \frac{72}{x} + 18 : 20 :: 72 + 18x : 20x;$$

$$\therefore 20x^2 + 120x = 18x^2 + 108x + 144;$$

by transposition, $2x^2 + 12x = 144$,

or $x^2 + 6x = 72$;

completing the square, $x^2 + 6x + 9 = 72 + 9 = 81$;

extracting the root, $x + 3 = \pm 9$,

and $x = 6$, or -12;

$\therefore$ *B* had 6 horses in the pasture at first, and $\frac{72}{x} + 18 = 30$ shillings, *per* week, was the price of the pasture.

25. An Upholsterer has two square carpets divided into square yards by the lines of the pattern. Now he observes, that if he subtracts from the number of squares in the smaller carpet the number of yards in the side of the other, the square of the remainder will exceed the difference of the number of squares in the smaller carpet, and the number of yards in its side, by 88. Also the difference of the lengths of the sides of the carpets is 6 feet. Required the size of each carpet.

Let $x =$ the number of yards in a side of the less;

$\therefore x + 2 =$ the number in a side of the greater,

and $(x^2 - x - 2)^2 = x^2 - x + 88 = x^2 - x - 2 + 90$;
by transposition, $(x^2 - x - 2)^2 - (x^2 - x - 2) = 90$;
completing the square, $(x^2 - x - 2)^2 - (x^2 - x - 2) + \frac{1}{4}$
$= 90 + \frac{1}{4} = \frac{361}{4}$;

$$\text{extracting the root, } x^2 - x - 2 - \frac{1}{2} = \pm\frac{19}{2},$$

and $x^2 - x = 12$, or -7, the former of which only will give a possible value of x; and ∴

$$\text{completing the square, } x^2 - x + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

$$\text{extracting the root, } x - \frac{1}{2} = \pm\frac{7}{2},$$

$$\text{and } x = 4, \text{ or } -3;$$

consequently the carpets contain 16 and 36 square yards, respectively.

26. A Man playing at hazard won at the first throw as much money as he had in his pocket; at the second throw he won 5 shillings more than the square root of what he then had; at the third throw he won the square of all he then had; and then he had £112. 16*s*. What had he at first?

Let $x =$ the number of shillings he had at first;
∴ $2x =$ the number he had after the first throw,
$\sqrt{2x} + 5 =$ the number won the second throw,
and $2x + \sqrt{2x} + 5 =$ the number he had after the second throw;
also $(2x + \sqrt{2x} + 5)^2 =$ the number won the third throw;

$$\therefore (2x + \sqrt{2x} + 5)^2 + (2x + \sqrt{2x} + 5) = 2256;$$

completing the square,

$$(2x + \sqrt{2x} + 5)^2 + (2x + \sqrt{2x} + 5) + \frac{1}{4}$$

$$= 2256 + \frac{1}{4} = \frac{9025}{4};$$

$$\text{extracting the root, } 2x + \sqrt{2x} + 5 + \frac{1}{2} = \pm\frac{95}{2};$$

$\therefore\ 2x + \sqrt{2x} = 42$, or -53, the latter of which gives impossible values of x, and $\therefore\ 2x + \sqrt{2x} = 42$, to answer the conditions of the problem;

completing the square, $2x + \sqrt{2x} + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4}$;

$$\text{extracting the root, } \sqrt{2x} + \frac{1}{2} = \pm\frac{13}{2};$$

$$\therefore\ \sqrt{2x} = 6, \text{ or } -7,$$

$$\text{and } 2x = 36, \text{ or } 49;$$

$$\therefore\ x = 18, \text{ or } \frac{49}{2};$$

and consequently he began with 18 shillings.

27. What number is that, which being divided by the product of its two digits, the quotient is two, and if 27 be added to it, the digits will be inverted?

Let x and y be the digits;

$$\therefore\ 10x + y = \text{the number},$$

$$\text{and } \frac{10x + y}{xy} = 2;$$

$$\therefore\ 10x + y = 2xy;$$

$$\text{also } 10x + y + 27 = 10y + x;$$

$$\text{by transposition, } 9x + 27 = 9y,$$

$$\text{or } x + 3 = y;$$

which value of y being substituted in the first equation,

$$10x + x + 3 = 2x\,.\,(x + 3),$$

$$\text{or } 11x + 3 = 2x^2 + 6x;$$

$$\text{by transposition, } 2x^2 - 5x = 3,$$

$$\text{or } x^2 - \frac{5}{2}\,.\,x = \frac{3}{2};$$

$$\text{completing the square, } x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{3}{2} + \frac{25}{16} = \frac{49}{16};$$

$$\text{extracting the root, } x - \frac{5}{4} = \pm\frac{7}{4};$$

and $x = 3$, or $-\frac{1}{2}$, the first of which only answers the conditions; $\therefore\ y = 6$, and the number is 36.

28. There are three numbers, the difference of whose differences is 8; their sum is 41; and the sum of their squares 699. What are the numbers?

Let $x =$ the second number,
and $y =$ the difference of the second and the least;
$\therefore\ x - y$, x, and $x + y + 8$ are the numbers,

$$\text{and their sum} = 3x + 8 = 41;$$
$$\text{by transposition, } 3x = 33,$$
$$\text{and } x = 11;$$
$$\therefore\ (11 - y)^2 + 121 + (19 + y)^2 = 699;$$
$$\text{or } 603 + 16y + 2y^2 = 699,$$
$$\text{by transposition, } 2y^2 + 16y = 96,$$
$$\text{and } y^2 + 8y = 48;$$
$$\text{completing the square, } y^2 + 8y + 16 = 64;$$
$$\text{extracting the root, } y + 4 = \pm 8,$$

and $y = 4$, or -12, both which values answer the conditions; and the numbers are 7, 11, and 23.

29. There are three numbers, the difference of whose differences is 5; their sum is 44; and continual product is 1950. What are the numbers?

Let $x =$ the second number,
and $y =$ the difference of the second and the least;
$\therefore$ the numbers are $x - y$, x, $x + y + 5$,

$$\text{and } (x - y + x + x + y + 5 =)\ 3x + 5 = 44;$$
$$\text{by transposition, } 3x = 39,$$
$$\text{and } x = 13;$$
$$\therefore\ (13 - y)\,.\,13\,.\,(18 + y) = 1950,$$
$$\text{and } (13 - y)\,.\,(18 + y) = 150,$$
$$\text{or } 234 - 5y - y^2 = 150;$$
$$\therefore\ y^2 + 5y = 84;$$
$$\text{completing the square, } y^2 + 5y + \frac{25}{4} = 84 + \frac{25}{4} = \frac{361}{4};$$

$$\text{extracting the root, } y + \frac{5}{2} = \pm \frac{19}{2},$$

and $y = 7$, or -12, both which values answer the conditions; $\therefore$ the numbers are 6, 13, and 25.

30. There were two rows of counters, of which the upper row exceeded the lower by one. A certain number having been taken from the upper row, and as many as then remained from the lower row, it was found that the square of the number remaining in the lower row, added to the square root of that number, is equal to 72 divided by the excess of the number taken from the upper row above unity. Required the number of counters taken from the upper row.

Let $x + 1 =$ the number in the upper row;
$\therefore x =$ the number in the lower;
$y + 1 =$ the number taken from the upper row;
$\therefore x - y =$ the remainder in the upper row,
and $y =$ the remainder in the lower row;

$$\therefore y^2 + \sqrt{y} = \frac{72}{y},$$

$$\text{and } y^3 + y^{\frac{3}{2}} = 72;$$

$$\text{completing the square, } y^3 + y^{\frac{3}{2}} + \frac{1}{4} = 72 + \frac{1}{4} = \frac{289}{4};$$

$$\text{extracting the root, } y^{\frac{3}{2}} + \frac{1}{2} = \pm \frac{17}{2};$$

$$\therefore y^{\frac{3}{2}} = 8, \text{ or } -9,$$

$$\text{and } y^3 = 64, \text{ or } 81;$$

$\therefore y = 4$, or $\sqrt[3]{81}$, the latter of which is excluded by the nature of the question; $\therefore$ the number required $(= y + 1) = 5$.

31. A Grocer sold 80 pounds of mace, and 100 pounds of cloves for £65; but he sold 60 pounds more of cloves for £20, than he did of mace for £10. What was the price of a pound of each?

Let $x =$ the price of a pound of mace (in pounds),
and $y =$ the price of a lb. of cloves;

then $x : 10 :: 1 :$ the number of lbs. of mace for £10 $= \frac{10}{x}$.

In the same way $\frac{20}{y} =$ the number of lbs. of cloves for £20;

$$\therefore \frac{20}{y} = 60 + \frac{10}{x},$$

$$\text{or } \frac{2}{y} = 6 + \frac{1}{x} = \frac{6x + 1}{x},$$

$$\text{and } \therefore y = \frac{2x}{6x + 1}$$

$$\text{Again, } 80x + 100y = 65,$$

$$\text{or } 16x + 20y = 13;$$

$$\therefore 16x + \frac{40x}{6x + 1} = 13,$$

$$\text{and } 96x^2 + 56x = 78x + 13;$$

$$\text{by transposition, } 96x^2 - 22x = 13,$$

$$\text{and } x^2 - \frac{22}{96} . x = \frac{13}{96};$$

$$\text{completing the square, } x^2 - \frac{22}{96} . x + \left(\frac{11}{96}\right)^2 = \frac{121}{(96)^2} + \frac{13}{96} = \frac{1369}{(96)^2};$$

$$\text{extracting the root, } x - \frac{11}{96} = \pm \frac{37}{96};$$

and $\therefore x = \frac{1}{2}$, or $-\frac{13}{48}$, which last does not answer the conditions; and $y = \frac{1}{4}$;

$\therefore$ the price of a pound of mace is 10 shillings, and of a pound of cloves is 5 shillings.

32. *A* and *B* engage to reap a field for £4. 10*s.*; and as *A* alone could reap it in 9 days, they promise to complete it in 5 days. They found however that they were obliged to call in *C*, an inferior workman, to assist them for the two last days, in consequence of which *B* received 3*s.* 9*d.* less

than he otherwise would have done. In what time could B or C alone reap the field?

Let x = the number of days in which B could reap the field, and y = the number in which C could reap it;

then $\frac{1}{9} + \frac{1}{x} : \frac{1}{x} :: 90$: the number of shillings B would have received $= \frac{810}{9+x}$,

and $\frac{5}{x} \times 90 = \frac{450}{x}$ = the number he did receive;

$$\therefore \frac{810}{9+x} - \frac{450}{x} = 3\tfrac{3}{4} = \frac{15}{4},$$

$$\text{or } \frac{54}{9+x} - \frac{30}{x} = \frac{1}{4};$$

$$\therefore 216x - 1080 - 120x = 9x + x^2;$$

by transposition, $x^2 - 87x = -1080$;

completing the square, $x^2 - 87x + \frac{7569}{4} = \frac{7569}{4} - 1080 = \frac{3249}{4}$;

extracting the root, $x - \frac{87}{2} = \pm\frac{57}{2}$,

and $x = 72$, or 15.

Let $x = 15$, then $\frac{5}{9} + \frac{5}{15} + \frac{2}{y} = 1$;

by transposition, $\frac{2}{y} = 1 - \frac{5}{9} - \frac{1}{3} = \frac{1}{9}$;

$\therefore y = 18$ the number of days in which C could reap the field. The other value of x is excluded by the nature of the question.

33. Throwing out the three court cards from a suit of spades, and placing the remainder in two heaps, I find the sum of the pips in the smaller heap is to the sum in the greater as the number of cards in the greater heap is to the number of cards in the smaller. But if I add the seventh card to the smaller heap, the difference of the number of pips in the two heaps is equal the square of the number of

cards in the smaller. Required the number of pips and cards in each.

The whole number of cards $= 10$,
and the whole number of pips $= 55$.

If $\therefore x =$ the number of cards in the larger heap;
$10 - x =$ the number in the smaller,
and if $y =$ the number of pips in the smaller;
$55 - y =$ the number in the larger;

$$\therefore y : 55 - y :: x : 10 - x,$$

and (*Alg.* 179, 3.) $y : 55 :: x : 10$;

$$\text{or } y : 11 :: x : 2;$$

$$\therefore 2y = 11x.$$

Again, after the change, $y + 7 =$ the number of pips in the smaller heap,

and $55 - y - 7 = 48 - y =$ the number of pips in the larger heap,

and $\therefore$ their difference $= 2y - 41 = (11 - x)^2$,

and by substituting for $2y$ its value, $11x - 41 = (11 - x)^2 = 121 - 22x + x^2$,

and $\therefore$ by transposition, $x^2 - 33x = - 162$;

completing the square, $x^2 - 33x + \left(\frac{33}{2}\right)^2 = \frac{1089}{4} - 162 = \frac{441}{4}$;

extracting the root, $x - \frac{33}{2} = \pm \frac{21}{2}$,

and $x = 6$, or 27; but 27 being inconsistent with the nature of the problem, the number of cards in the larger heap $= 6$, and $\therefore$ the number in the smaller heap $= 4$; and the number of pips in the smaller $= \frac{11x}{2} = 33$, and $\therefore$ the number in the greater heap $= 22$.

34. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each.

Let x = number of yards in circumference of the larger, and y = the number in the circumference of the less;

$$\text{then } \frac{120}{x} = \frac{120}{y} - 6,$$

$$\text{or } 20y = 20x - xy;$$

$$\therefore \text{ by transposition, } xy = 20x - 20y.$$

$$\text{Again, } \frac{120}{x+1} = \frac{120}{y+1} - 4,$$

$$\text{or } 30 \cdot (y+1) = (x+1) \cdot (29 - y),$$

$$\text{or } 30y + 30 = 29x + 29 - xy - y;$$

$$\text{by transposition, } xy = 29x - 1 - 31y,$$

$$\text{and } \therefore 29x - 1 - 31y = 20x - 20y;$$

$$\text{by transposition, } 9x = 11y + 1,$$

$$\text{or } x = \frac{11y+1}{9};$$

$$\therefore \text{ by substitution, } \frac{11y^2 + y}{9} = \frac{220y + 20}{9} - 20y,$$

$$\text{or } 11y^2 + y = 220y + 20 - 180y = 40y + 20;$$

$$\therefore y^2 - \frac{39}{11} \cdot y = \frac{20}{11};$$

completing the square, $y^2 - \frac{39}{11} \cdot y + \left(\frac{39}{22}\right)^2 = \frac{1521}{(22)^2} + \frac{20}{11} = \frac{2401}{(22)^2}$;

$$\text{extracting the root, } y - \frac{39}{22} = \pm \frac{49}{22};$$

$$\therefore y = 4, \text{ or } -\frac{5}{11};$$

$\therefore$ the number of yards in the circumference of the less = 4, and the number in the circumference of the greater

$$= \frac{11y+1}{9} = 5.$$

35. On the late jubilee, a gentleman treated his tenantry at the following rate. He allowed for each poor child a certain number of sixpences, for each poor woman sixpence more,

and for each poor man sixpence still in addition. The number of women was one-fourth greater than the number of men; the number of children was equal to twice the square of the difference between the numbers of men and women; and the whole expense was £8. 2s. But had each child been allowed as much as each woman, the expense on their account added to nine times the difference of what the men and women cost, would have been £4. 18s. Required the number of men, women, and children, and the allotment to each.

Let $4x =$ the number of men;
$\therefore\ 5x =$ the number of women,
and $2x^2 =$ the number of children.

Let $y =$ the number of sixpences each child had;
$\therefore\ y + 1 =$ the number each woman had;
and $y + 2 =$ the number each man had;

$$\therefore\ 2x^2y + 9xy + 13x = 324;$$
$$\text{also } 2x^2y + 2x^2 + 9xy - 27x = 196;$$
$$\therefore \text{ by subtraction, } 2x^2 - 40x = -128,$$
$$\text{and } x^2 - 20x = -64;$$
$$\text{completing the square, } x^2 - 20x + 100 = 100 - 64 = 36;$$
$$\text{extracting the root, } x - 10 = \pm 6,$$

and $x = 16$, or 4, the former of which will not answer the conditions of the problem; $\therefore$ the number of men was 16, of women 20, and of children 32.

$$\text{Also } 32y + 36y + 52 = 324,$$
$$\text{or } 68y = 272;$$
$$\therefore\ y = 4;$$

$\therefore$ each child had 2 shillings, each woman 2s. 6d., and each man 3s.

36. *A* and *B* were going to market, the first with cucumbers, and the second with three times as many eggs; and they find that if *B* gave all his eggs for the cucumbers, *A* would lose 10 pence, according to the rate at which they were then selling. *A* therefore reserves two-fifths of his cucumbers; by which *B* would lose sixpence, according to

the same rate. But B, selling the cucumbers at sixpence apiece, gains upon the whole the price of six eggs. Required the number of eggs and cucumbers, and their price.

Let $x =$ the number of cucumbers,
and $y =$ the price of one;
$\therefore\ 3x =$ the number of eggs,

and $\dfrac{xy-10}{3x} =$ the price of one egg;

$$\text{also } \frac{3}{5}xy = xy - 16,$$

$$\text{or } 3xy = 5xy - 80;$$

$$\therefore\ 2xy = 80,$$

$$\text{and } xy = 40.$$

$$\text{Also } \frac{3}{5}\,.\,6x - (xy - 10) = \frac{2\,.\,(xy-10)}{x},$$

$$\text{or } \frac{18}{5}x^2 - 30x = 60,$$

$$\text{and } 9x^2 - 75x = 150;$$

completing the square, $9x^2 - 75x + \left(\dfrac{25}{2}\right)^2 = \dfrac{8625}{4} + 150$
$= \dfrac{1225}{4}$;

$$\text{extracting the root, } 3x - \frac{25}{2} = \pm\frac{35}{2},$$

and $3x = 30$, or -5, the latter of which is excluded by the nature of the problem; $\therefore\ x = 10$,

$$\text{and } y = \frac{40}{x} = 4.$$

Hence the number of eggs was 30, and of cucumbers 10;
$\therefore$ the price of a cucumber was 4 pence, and of an egg $= \dfrac{xy-10}{3x} = 1$ penny.

37. A person bought a certain number of larks and sparrows for 6 shillings. He gave as many pence *per* dozen for larks as there were sparrows, and as many pence *per* score for

sparrows as there were larks. If he had bought 10 more of each, (the price of larks remaining the same,) and had given as much *per* dozen for sparrows as he gave *per* score for larks, they would have cost £1. 5*s*. 5*d*. Required the number of each.

Let $x =$ the number of larks, and $\therefore =$ the number of pence *per* score for sparrows,

$y =$ the number of sparrows, and $\therefore =$ the number of pence *per* dozen for larks;

$$\therefore \left(\frac{xy}{12} + \frac{xy}{20} =\right) \frac{2xy}{15} = 72,$$

$$\text{and } xy = 15 \times 36 = 540.$$

Again, if $x + 10 =$ the number of larks,

and $y + 10 =$ the number of sparrows;

then the price of the larks $= y \times \dfrac{x + 10}{12} = \dfrac{xy + 10y}{12}$

$= 10 \cdot \dfrac{54 + y}{12}$,

and the price *per* dozen of sparrows $= \dfrac{y}{12} \times 20 = \dfrac{5y}{3}$;

$$\therefore \text{ the price of the sparrows} = \frac{5y^2 + 50y}{3 \times 12},$$

$$\text{and } \frac{54 + y}{12} \times 10 + \frac{5 \cdot (y^2 + 10y)}{3 \times 12} = 305,$$

$$\text{and } (54 + y)\, 30 + 5 \cdot (y^2 + 10y) = 36 \times 305,$$

$$\text{or } y^2 + 16y + 324 = 36 \times 61 = 2196;$$

by transposition, $y^2 + 16y = 1872$;

completing the square, $y^2 + 16y + 64 = 1872 + 64 = 1936$;

extracting the root, $y + 8 = \pm 44$;

$\therefore y = 36$, or -52, the latter of which will not answer the conditions of the problem,

$$\text{and } x = \frac{15 \times 36}{y} = 15;$$

$\therefore$ he bought 15 larks, and 36 sparrows.

38. A Poulterer bought a certain number of ducks and 18 turkeys for £5. 10*s*.; each turkey costing within one shil-

ling as much as three ducks. He afterwards bought as many ducks and 5 over, and 20 turkeys, giving one shilling a piece more for each duck and turkey than before; and found that the value of his former purchase was to the value of the latter one :: 2 : 3. Required the number of ducks, and the prices of the ducks and turkeys at the first purchase.

Let $x =$ the number of ducks required,
and $y =$ the price of a duck;
$\therefore\ 3y - 1 =$ the price of a turkey,

$$\text{and } xy + 54y - 18 = 110,$$
$$\text{or } xy + 54y = 128.$$

Now at the second purchase, $x + 5 =$ the number of ducks, $y + 1$, and $3y =$ the price of a duck and turkey, respectively;

$$\therefore\ 110 : xy + x + 65y + 5 :: 2 : 3,$$
$$\text{or } 55 : xy + x + 65y + 5 :: 1 : 3;$$
$$\therefore\ xy + x + 65y + 5 = 165,$$
$$\text{and } xy + x + 65y = 160;$$
$$\text{but } xy + 54y = 128;$$
$$\therefore \text{ by subtraction, } x + 11y = 32,$$

or $x = 32 - 11y$, which being substituted in the first equation,

$$32y - 11y^2 + 54y = 128,$$
$$\text{or } 11y^2 - 86y = -128,$$
$$\text{and } y^2 - \frac{86}{11}.y = -\frac{128}{11};$$

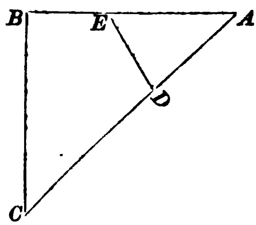
completing the square, $y^2 - \frac{86}{11}.y + \left(\frac{43}{11}\right)^2 = \frac{1849}{121} - \frac{128}{11} = \frac{441}{821};$

$$\text{extracting the root, } y - \frac{43}{11} = \pm\frac{21}{11},$$

and $y = \frac{64}{11}$, or 2, the former of which makes x negative, and $\therefore$ the price of a duck is 2 shillings, and the price of a turkey $= 5$ shillings. Also the number of ducks $= 32 - 11y = 10.$

39. There are three towns, A, B, and C; the road from B to A forming a right angle with that from B to C. Now a person has to go from B to A, but after travelling a certain distance towards A, he crosses over by the nearest way to the road which leads from C to A, and when on this road he is 3 miles from A and 7 from C. He then proceeds to A, and when arrived there he finds that he has gone a distance, equal to one-fourth of the distance from B to C, more than he would have done, had he gone the direct road from B to A. Required the distance of B from A and C.

$$\text{Let } BC = x,\ BA = y,\ AC = 10,$$



and since the shortest path from a given point to a given straight line is a perpendicular drawn from that point, draw ED perpendicular to AC; E being the point where he leaves the road BA; $\therefore$ D is the point where he enters the road CA; $\therefore$ $CD = 7$, and $DA = 3$.

$$\text{By similar } \Delta\text{'s } BA : CA :: DA : AE,$$

$$\text{or } y : 10 :: 3 : AE = \frac{30}{y},$$

$$\text{and } BA : BC :: DA : DE,$$

$$\text{or } y : x :: 3 : DE = \frac{3x}{y};$$

$$\therefore\ \frac{3x}{y} + 3 = \frac{30}{y} + \frac{x}{4},$$

$$\text{or } 3 \,.\, (x + y) = 30 + \frac{1}{4}xy,$$

$$\text{and } 2xy - 24 \,.\, (x + y) = -240;$$

but $x^2 + y^2 = 100$;

$\therefore$ by addition, $x^2 + 2xy + y^2 - 24 \cdot (x + y) = -140$;

completing the square, $(x + y)^2 - 24 \cdot (x + y) + 144 = 144 - 140 = 4$;

extracting the root, $x + y - 12 = \pm 2$;

$\therefore x + y = 14$, or 10, the former of which only answers the conditions, $\therefore x^2 + 2xy + y^2 = 196$,

but $2x^2 \qquad + 2y^2 = 200$;

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 4$,

whence $x - y = \pm 2$,

but $x + y = 14$;

$\therefore$ by addition, $2x = 16$, or 12,

and by subtraction, $2y = 12$, or 16;

$\therefore x = 8$, or 6,

and $y = 6$, or 8;

$\therefore$ the distance of *B* from *C* is 8, or 6 miles, and from *A* is 6, or 8 miles.

40. In a garden is a square bowling-green, a side of which is 30 yards, and near to it is a rectangular grass-plot. The number of square yards in the area of the grass-plot is a mean proportional between $\frac{192}{79}$, and the number of square yards contained in the grass-plot and bowling-green together. Also the number of square yards contained in the square described on the diameter of the grass-plot is a mean proportional between 10, and the number of square yards contained in the aforesaid square increased by the number contained in the bowling-green. Required the area and sides of the grass-plot.

Let x and y = the number of yards in the sides;

$\therefore xy$ = the area,

$$\text{and } \frac{192}{79} : xy :: xy : xy + 900;$$

$$\therefore x^2y^2 = \frac{192}{79} \cdot xy + \frac{172800}{79};$$

by transposition, $x^2y^2 - \frac{192}{79}xy = \frac{172800}{79}$;

completing the square, $x^2y^2 - \frac{192}{79}xy + \left(\frac{96}{79}\right)^2 = \frac{172800}{79} + \left(\frac{96}{79}\right)^2$

$= \frac{13660416}{(79)^2}$;

extracting the root, $xy - \frac{96}{79} = \pm \frac{3696}{79}$;

$\therefore$ $xy = 48$, or $-\frac{3600}{79}$, the former of which only answers the conditions of the problem.

Again, $10 : x^2 + y^2 :: x^2 + y^2 : x^2 + y^2 + 900$;

$\therefore$ $(x^2 + y^2)^2 = 10 \,.\, (x^2 + y^2) + 9000$;

by transposition, $(x^2 + y^2)^2 - 10 \,.\, (x^2 + y^2) = 9000$;

completing the square, $(x^2 + y^2)^2 - 10 \,.\, (x^2 + y^2) + 25 = 9025$;

extracting the root, $x^2 + y^2 - 5 = 95$;

$\therefore$ $x^2 + y^2 = 100$, or -90, the latter of which will not answer the conditions;

but, $2xy = 96$;

$\therefore$ by addition, $x^2 + 2xy + y^2 = 196$, and $x + y = \pm 14$;

by subtraction, $x^2 - 2xy + y^2 = 4$, and $x - y = \pm 2$;

$\therefore$ by addition, $2x = \pm 16$, or ± 12;

$\therefore$ $x = \pm 8$, or ± 6;

by subtraction, $2y = \pm 12$, or ± 16;

$\therefore$ $y = \pm 6$, or ± 8;

but the positive values will only answer the conditions, and the area is 48 square yards.

41. A rectangular vat, 3 feet deep, when filled to the depth of 2 feet, holds less than when completely filled by a number of cubic feet equal to 24, together with half the number of feet in the perimeter of the base. It is also observed, that the length of a pole, which reaches from one of the corners of the top to the opposite corner of the bottom of the vat, is equal to one-eighth of the number of feet in the square inscribed on the diagonal of the bottom. Required the dimensions of the vat.

Let x and y be the number of feet in the sides of the base;

$$\text{then } (3xy - 2xy =)\ xy = 24 + x + y,$$

$$\text{and } \sqrt{(9 + x^2 + y^2)} = \frac{1}{8} \cdot (x^2 + y^2);$$

$$\therefore\ x^2 + y^2 - 8\sqrt{(9 + x^2 + y^2)} = 0,$$

$$\text{and } (9 + x^2 + y^2) - 8\sqrt{(9 + x^2 + y^2)} = 9;$$

completing the square, $(9 + x^2 + y^2) - 8\sqrt{(9 + x^2 + y^2)} + 16 = 25;$

extracting the root, $\sqrt{(9 + x^2 + y^2)} - 4 = \pm 5,$

$$\text{and } \sqrt{(9 + x^2 + y^2)} = 9, \text{ or } -1;$$

$$\therefore\ 9 + x^2 + y^2 = 81, \text{ or } 1;$$

by transposition, $x^2 + y^2 = 72$, or -8, the latter of which is impossible;

$$\text{but } 2xy = 48 + 2 \cdot (x + y);$$

$\therefore$ by addition, $x^2 + 2xy + y^2 = 120 + 2 \cdot (x + y);$

by transposition, $(x + y)^2 - 2 \cdot (x + y) = 120;$

completing the square, $(x + y)^2 - 2 \cdot (x + y) + 1 = 121;$

extracting the root, $x + y - 1 = \pm 11,$

and $x + y = 12$, or -10, the latter of which is impossible;

$$\therefore\ x^2 + 2xy + y^2 = 144;$$

$$\text{but } 4xy = 144;$$

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 0,$

$$\text{and } x - y = 0;$$

$$\text{but } x + y = 12;$$

$\therefore$ by addition, $2x = 12,$

$$\text{and } x = 6;$$

$\therefore\ y = x = 6$, and the base is a square whose side is 6 feet.

42. A person bought two cubical stacks of hay for £41, each of which cost as many shillings *per* solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each?

Let x = the number of yards in a side of the larger,
and y = the number in a side of the less;

then x^3 and y^3 = the number of solid yards in the stacks, and x^2 and y^2 = the number of square yards in their bases;

$$\therefore\ x^2 - y^2 = 9,$$

$$\text{and } x^2y + y^2x = 820;$$

$$\therefore\ x^2 + y^2 = \frac{820}{xy},$$

$$\text{and } x^4 + 2x^2y^2 + y^4 = \frac{(820)^2}{x^2y^2};$$

$$\text{but } x^4 - 2x^2y^2 + y^4 = 81;$$

$$\therefore \text{ by subtraction, } 4x^2y^2 = \frac{(820)^2}{x^2y^2} - 81,$$

$$\text{or } x^4y^4 = \frac{(820)^2}{4} - \frac{81x^2y^2}{4};$$

$$\text{by transposition, } x^4y^4 + \frac{81}{4}\,.\,x^2y^2 = \left(\frac{820}{2}\right)^2 = (410)^2 = 168100;$$

$$\text{completing the square, } x^4y^4 + \frac{81}{4}\,.\,x^2y^2 + \left(\frac{81}{8}\right)^2 = 168100 + \frac{6561}{64} = \frac{10764961}{64};$$

$$\text{extracting the root, } x^2y^2 + \frac{81}{8} = \pm\frac{3281}{8};$$

$\therefore\ x^2y^2 = 400$, or $-\dfrac{1681}{4}$, the latter of which is impossible;

$\therefore\ xy = \pm 20$, the positive value only answering the conditions of the problem;

$$\text{and } x^2 + y^2 = \frac{820}{xy} = \frac{820}{20} = 41;$$

$$\text{but } 2xy = 40;$$

$\therefore$ by addition, $x^2 + 2xy + y^2 = 81$, and $x + y = \pm 9$,

and by subtraction, $x^2 - 2xy + y^2 = 1$; $\therefore\ x - y = \pm 1$;

$\therefore$ by addition, $2x = \pm 10$, or ± 8,

and $x = \pm 5$, or ± 4;

by subtraction, $2y = \pm 8$, or ± 10,

and $y = \pm 4$, or ± 5;

$\therefore$ the prices were £25, and £16.

43. *A* and *B* put out different sums to interest, amounting together to £200. *B*'s rate of interest was £1 *per* cent. more than *A*'s. At the end of 5 years, *B*'s accumulated simple interest wanted but £4 to be double of *A*'s. At the end of 10 years, *A*'s principal and interest was to *B*'s as 5 : 8. Required the separate sums put out by each, and the rate *per* cent.

Let $4x = A$'s money (in pounds);

$\therefore\ 4 \,.\, (50 - x) = B$'s money;

$y = A$'s rate of interest;

$\therefore\ y + 1 = B$'s rate;

$\therefore\ \frac{xy}{5} = A$'s interest after 5 years,

and $\frac{(50 - x) \,.\, (y + 1)}{5} = B$'s interest after 5 years;

$$\therefore\ \frac{(50 - x) \,.\, (y + 1)}{5} + 4 = 2 \,.\, \frac{xy}{5},$$

$$\text{or } 50y - xy + 50 - x + 20 = 2xy;$$

$$\therefore\ 3xy = 50y + 70 - x \quad . \quad . \quad . \quad (1).$$

Again, after 10 years, *A*'s capital and interest =

$$4x + \frac{2xy}{5} = 2 \,.\, \frac{10x + xy}{5},$$

$$\text{and } B\text{'s} = 2 \,.\, \frac{(50 - x) \,.\, (y + 11)}{5}$$

$$\therefore\ 2 \,.\, \frac{10x + xy}{5} : 2 \,.\, \frac{(50 - x) \,.\, (y + 11)}{5} :: 5 : 8;$$

$$\therefore\ 80x + 8xy = 250y - 5xy + 2750 - 55x;$$

by transposition, $13xy = 250y + 2750 - 135x$. (2)

but from (1) $15xy = 250y + 350 - 5x$;

$\therefore$ by subtraction, $2xy = 130x - 2400$ . . . (3)

multiplying (1) by 13, and (2) by 3;

$$\therefore\ 650y + 910 - 13x = (39xy =)\ 750y + 8250 - 405x;$$

by transposition, $100y = 392x - 7340$,

or $50y = 196x - 3670$, which being multiplied by x,

$50xy = 196x^2 - 3670x$; but (3) being multiplied by 25,

$$50xy = 3250x - 60000;$$

$$\therefore\ 196x^2 - 3670x = 3250x - 60000;$$

by transposition, $196x^2 - 6920x = -60000;$

completing the square, $196x^2 - 6920x + \left(\frac{1730}{7}\right)^2$

$$= \frac{2992900}{49} - 60000 = \frac{52900}{49};$$

extracting the root, $14x - \frac{1730}{7} = \pm\frac{230}{7};$

$$\therefore\ 14x = 280, \text{ or } \frac{1500}{7};$$

$$\therefore\ x = 20;$$

$\therefore$ *A*'s money $= 4x = 80$ pounds, and *B*'s $= 120$ pounds,

$$\text{and } y = \frac{196 \times 20 - 3670}{50} = 5;$$

$\therefore$ *A*'s rate of interest was 5 *per* cent. and *B*'s 6 *per* cent.

44. When the price of brandy was three times the price of British spirit, a merchant made two mixtures of brandy and British spirit, and the prices *per* gallon were in the ratio of 9 to 10. He afterwards mixed twice as much brandy with the same quantity of British spirit in each case, and the relative price was the same as before. Required the ratio of the quantities mixed.

Suppose at first x gallons of British spirit were mixed with one gallon of brandy in one case; and y gallons of British spirit with one gallon of brandy in the other case; then

$$\left.\begin{aligned} x + 1 : 1 :: x + 3 &: \frac{x+3}{x+1} \\ \text{and } y + 1 : 1 :: y + 3 &: \frac{y+3}{y+1} \end{aligned}\right\} \text{the relative prices of the first mixtures } \textit{per} \text{ gallon;}$$

$$\text{hence, } \frac{x+3}{x+1} : \frac{y+3}{y+1} :: 9 : 10.$$

$$\text{In the same manner } \frac{x+6}{x+2} : \frac{y+6}{y+2} :: 9 : 10,$$

and $\therefore\ xy + x + 3y + 3 : xy + 3x + y + 3 :: 9 : 10,$

div^{do}. $xy + x + 3y + 3 : 2x - 2y :: 9 : 1;$

$$\therefore xy + x + 3y + 3 = 18x - 18y;$$

by transposition, $xy - 17x + 21y + 3 = 0;$

also $xy + 2x + 6y + 12 : xy + 6x + 2y + 12 :: 9 : 10;$

div^{do}. $xy + 2x + 6y + 12 : 4x - 4y :: 9 : 1;$

$$\therefore xy + 2x + 6y + 12 = 36x - 36y;$$

by transposition, $xy - 34x + 42y + 12 = 0;$

but $xy - 17x + 21y + 3 = 0;$

$\therefore$ by subtraction, $17x - 21y - 9 = 0;$

also $2xy - 34x + 42y + 6 = 0,$

and $xy - 34x + 42y + 12 = 0;$

$\therefore$ by subtraction, $xy - 6 = 0,$

or $xy = 6.$

$$\text{Now } x = \frac{21y + 9}{17};$$

$$\therefore \frac{21y^2 + 9y}{17} = (xy =)\ 6,$$

and $21y^2 + 9y = 102,$

$$\text{or } y^2 + \frac{3}{7}y = \frac{34}{7};$$

completing the square, $y^2 + \frac{3}{7}y + \frac{9}{196} = \frac{34}{7} + \frac{9}{196} = \frac{961}{196};$

extracting the root, $y + \frac{3}{14} = \pm\frac{31}{14};$

$$\therefore y = 2, \text{ or } -\frac{17}{7};$$

$$\therefore x = 3;$$

$\therefore$ the first mixtures were in the ratio of 3 to 1, and 2 to 1; and the second in the ratio of 3 to 2, and of equality.

SECTION X.

Examples of the Solution of Problems in Arithmetical and Geometrical Progressions.

1. A Person bought 7 books, the particular prices of which (in shillings) were in arithmetical progression. The price of the next above the cheapest was 8 shillings, and the price of the dearest 23 shillings. What was the price of each book?

$$\text{Let } x = \text{the price of the cheapest,}$$
$$\text{and } y = \text{the common difference;}$$
$$\text{then } x + y = \text{the price of the second} = 8,$$
$$\text{and } x + 6y = \text{the price of the dearest} = 23;$$
$$\therefore \text{ by subtraction, } 5y = 15,$$
$$\text{and } y = 3;$$
$$\therefore x = 8 - y = 8 - 3 = 5,$$

and $\therefore$ the prices are 5, 8, 11, 14, 17, 20, 23 shillings, respectively.

2. A number consists of 3 digits, which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits will be inverted. Required the number.

$$\text{Let } \left.\begin{matrix} x - y \\ x \\ x + y \end{matrix}\right\} \text{ represent the digits;}$$

then the number will be $100 . (x - y) + 10x + x + y = 111x - 99y$;

$$\therefore \frac{111x - 99y}{3x} = 26,$$
$$\text{or } 37x - 33y = 26x,$$
$$\text{and } \therefore \text{ by transposition, } 11x = 33y,$$
$$\text{and } x = 3y.$$

Again, $111x - 99y + 198 = 100 \cdot (x + y) + 10x + (x - y) = 111x + 99y$;

$\therefore$ by transposition, $198y = 198$,

and $y = 1$;

$\therefore x = 3y = 3$;

$\therefore$ the digits are 2, 3, and 4, and the number = 234.

3. The sum of £1. 7*s.* was to be raised by subscription by three persons *A*, *B*, and *C*; the sums to be subscribed by them respectively forming an arithmetical progression. But *C* dying before the money was paid, the whole fell to *A* and *B*; and *C*'s share was raised between them in the proportion of 3 : 2, when it appeared that the whole sum subscribed by *A* was to the whole sum subscribed by *B* :: 4 : 5. Required the original subscriptions of *A*, *B*, and *C*.

Let $x - y$, x, $x + y$, be the respective subscriptions of *A*, *B*, and *C*;

then $3x = 27$; and $\therefore x = 9$.

Now $5 : 2 ::$ (*C*'s share =) $9 + y$: the part paid by $B = \frac{2}{5} \cdot (9 + y)$,

and $5 : 3 :: 9 + y$: the part paid by $A = \frac{3}{5} \cdot (9 + y)$,

and consequently, *A* paid upon the whole $9 - y + \frac{3}{5} \cdot (9 + y) = \frac{72 - 2y}{5}$;

also *B* paid upon the whole $9 + \frac{2}{5}(9 + y) = \frac{63 + 2y}{5}$;

hence, $72 - 2y : 63 + 2y :: 4 : 5$,

and (*Alg.* 179, 3.) $135 : 63 + 2y :: 9 : 5$,

and (*Alg.* 179, 8.) $15 : 63 + 2y :: 1 : 5$;

$\therefore$ (21) $63 + 2y = 75$;

by transposition, $2y = 12$,

and $y = 6$;

$\therefore$ the sums to be subscribed originally were 3, 9, and 15 shillings.

4. Four numbers are in arithmetical progression. The sum of their squares is equal to 276, and the sum of the numbers themselves is equal to 32. What are the numbers?

Let $2y =$ the common difference,

$$\left.\begin{aligned} \text{and } x &+ 3y \\ x &+ y \\ x &- y \\ x &- 3y \end{aligned}\right\} \text{be the numbers;}$$

$$\text{then their sum} = 4x = 32,$$

$$\text{and } \therefore x = 8;$$

also the sum of their squares $= 4x^2 + 20y^2 = 276$, in which substituting the value of x found above,

$$256 + 20y^2 = 276;$$

$$\text{by transposition, } 20y^2 = 20;$$

$$\therefore y^2 = 1;$$

$$\text{and } y = \pm 1;$$

hence the numbers are 11, 9, 7, 5.

5. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Supposing as before, $x + 3y$, $x + y$, $x - y$, and $x - 3y$, to be the numbers;

$$\text{then } 2x^2 + 18y^2 = 200,$$

$$\text{and } 2x^2 + 2y^2 = 136;$$

$$\therefore \text{by subtraction, } 16y^2 = 64,$$

$$\text{and } 4y = \pm 8;$$

$$\therefore y = \pm 2;$$

$$\text{whence } x^2 = 68 - y^2 = 68 - 4 = 64,$$

$$\text{and } x = \pm 8,$$

and $\therefore$ the numbers are ± 14, ± 10, ± 6, ± 2.

6. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers.

Let x, xy, xy^2, xy^3, be the numbers;

$$\therefore\ x + xy = 15,$$
$$\text{and } xy^2 + xy^3 = 60,$$
$$\text{or } y^2 \,.\, (x + xy) = 60,$$
$$\text{or } 15y^2 = 60;$$
$$\therefore\ y^2 = 4,$$
$$\text{and } y = \pm 2,$$
$$\text{and } (x + 2x =)\ 3x = 15;$$
$$\therefore\ x = 5,$$
and the numbers are 5, 10, 20, 40.

7. The sum of four numbers in geometrical progression is equal to the common ratio $+ 1$; and the first term $= \frac{1}{17}$. Required the numbers.

Let $x =$ the common ratio;

$$\therefore \text{ the numbers are } \frac{1}{17}, \frac{x}{17}, \frac{x^2}{17}, \frac{x^3}{17},$$
$$\text{and } 1 + x = \frac{1 + x + x^2 + x^3}{17} = \frac{(1 + x^2) \,.\, (1 + x)}{17},$$
$$\text{and } 1 = \frac{1 + x^2}{17};$$
$$\therefore\ 17 = 1 + x^2,$$
$$\text{and } 16 = x^2;$$
$$\therefore\ \pm 4 = x,$$
$$\text{and the numbers are } \frac{1}{17}, \frac{4}{17}, \frac{16}{17}, \frac{64}{17}.$$

8. A regiment of militia was just sufficient to form an equilateral wedge. It was afterwards doubled by the supplementary, but was still found to want 385 men to complete a square containing 5 more men in a side, than in a side of the wedge. How many did the regiment at first contain?

Let $x =$ the number of men in a side of the wedge;

$\therefore$ (*Alg.* 192.) $(x + 1) \,.\, \frac{x}{2} =$ the number of men in the wedge;

$$\therefore (x + 1) \cdot x + 385 = (x + 5)^2,$$
$$\text{or } x^2 + x + 385 = x^2 + 10x + 25;$$
$$\therefore \text{by transposition, } 360 = 9x,$$
$$\text{and } 40 = x;$$
$$\therefore \text{the number of men} = 820.$$

9. After A, who travelled at the rate of 4 miles an hour, had been set out two hours and three-quarters, B set out to overtake him, and in order thereto went four miles and a half the first hour, four and three-quarters the second, five the third; and so on, gaining a quarter of a mile every hour. In how many hours would he overtake A?

Let x = the number of hours;

$\therefore$ (*Alg.* 192.) $\left(9 + (x - 1)\frac{1}{4}\right) \times \frac{x}{2}$ = the whole number of miles he travelled;

but $11 + 4x$ = the whole number A travelled;

$$\therefore \left(9 + \frac{1}{4}(x - 1)\right) \cdot \frac{x}{2} = 11 + 4x,$$

$$\text{or } 9x + \frac{x^2}{4} - \frac{x}{4} = 22 + 8x,$$

$$\text{and by transposition, } \frac{x^2}{4} + \frac{3x}{4} = 22;$$

$$\text{completing the square, } \frac{x^2}{4} + \frac{3x}{4} + \frac{9}{16} = 22 + \frac{9}{16} = \frac{361}{16};$$

$$\text{extracting the root, } \frac{x}{2} + \frac{3}{4} = \pm\frac{19}{4};$$

$$\therefore \frac{x}{2} = 4, \text{ or } -\frac{11}{2};$$

$$\therefore x = 8, \text{ or } -11;$$

hence in 8 hours he would overtake him. -11 not answering the conditions of the problem.

10. The base of a right-angled triangle is 6, and the sides are in arithmetical progression; it is required to find the other two sides.

Let $6 - x$, 6, and $6 + x$ be the sides;
then $36 - 12x + x^2 + 36 = 36 + 12x + x^2$
(*Eucl.* B. I. *p.* 48.)
by transposition, $24x = 36$,

and $x = \frac{3}{2}$;

$\therefore$ the sides are $\frac{9}{2}$, $\frac{12}{2}$, and $\frac{15}{2}$.

But if 6 be the first term of the progression; let,
6, $6 + x$, $6 + 2x$ be the sides;
then $36 + 24x + 4x^2 = 36 + 36 + 12x + x^2$;
by transposition, $3x^2 + 12x = 36$,
or $x^2 + 4x = 12$;
completing the square, $x^2 + 4x + 4 = 16$;
extracting the root, $x + 2 = \pm 4$,
and $x = 2$, or -6,
and the sides are 6, 8, and 10; or 0, 6, and -6.

The problem is not properly restricted; the algebraical expression, in this instance, is more precise than the language in which the problem is stated.

11. *A* and *B* set out from London at the same time, to go round the world (23661 miles), one going East, the other West. *A* goes one mile the first day, two the second, and so on. *B* goes 20 miles a day. In how many days will they meet; and how many miles will be travelled by each?

Let x = the number of days;

then (*Alg.* 192.) $(x + 1) \cdot \frac{x}{2}$ = the number of miles *A* goes,

and $20x$ = the number *B* goes;

$$\therefore \frac{x^2 + x}{2} + 20x = 23661,$$

and $x^2 + 41x = 47322$;

$$\text{completing the square, } x^2 + 41x + \left(\frac{41}{2}\right)^2 = 47322 + \frac{1681}{4}$$
$$= \frac{190969}{4};$$

$$\therefore \text{ extracting the root, } x + \frac{41}{2} = \pm\frac{437}{2};$$

$$\therefore x = 198, \text{ or } -239;$$

$\therefore$ they travel 198 days; A goes 19701, and B 3960 miles.

12. A traveller sets out for a certain place, and travels one mile the first day, two the second, and so on. In 5 days afterwards another sets out, and travels 12 miles a day. How long and how far must he travel to overtake the first?

$$\text{Let } x = \text{the number of days;}$$
$$\text{then } x + 5 = \text{the number the first travels,}$$
$$\text{and} \therefore (Alg.\ 192.)\ (x + 6) \cdot \frac{x+5}{2} = \text{the distance he travels,}$$
$$\text{and } 12x = \text{the distance the second travels;}$$
$$\therefore (x + 6) \cdot \frac{x+5}{2} = 12x,$$
$$\text{and } x^2 + 11x + 30 = 24x;$$
$$\therefore \text{ by transposition, } x^2 - 13x = -30;$$
$$\text{completing the square, } x^2 - 13x + \frac{169}{4} = \frac{169}{4} - 30 = \frac{49}{4};$$
$$\text{extracting the root, } x - \frac{13}{2} = \pm\frac{7}{2};$$
$$\text{and } x = 3, \text{ or } 10.$$

$\therefore$ they are together at the end of 3, and 10 days after the second sets out; and 36 and 120 miles is the distance travelled.

13. A and B, 165 miles distant from each other, set out with a design to meet; A travels one mile the first day, two the second, three the third, and so on; B travels 20 miles the first day, 18 the second, 16 the third, and so on. How soon will they meet?

Let $x =$ the number of days required;

then $1 + 2 + 3 + \ldots + x = (1 + x) \cdot \frac{x}{2} =$ the number of miles A travelled,

and $20 + 18 + \ldots + 20 - 2x + 2 = (42 - 2x) \cdot \frac{x}{2} =$ the number B travelled;

$$\therefore (42 - 2x) \cdot \frac{x}{2} + (1 + x) \cdot \frac{x}{2} = (43 - x) \cdot \frac{x}{2} = 165,$$

$$\text{or } x^2 - 43x = -330;$$

completing the square, $x^2 - 43x + \frac{1849}{4} = \frac{1849}{4} - 330 = \frac{529}{4}$;

extracting the root, $x - \frac{43}{2} = \pm \frac{23}{2}$;

$$\therefore x = 10, \text{ or } 33.$$

Hence it appears that they meet in 10 days. On the 10th day B travels 2 miles, and the next day he rests; the following day he returns 2 miles; the succeeding day 4, and so on, increasing two miles every day; and on the 33d day he again comes up with A, who has been travelling forward, every day's journey being one mile longer than that of the preceding day.

14. There are four numbers in arithmetical progression whose continual product is 1680, and common difference is 4. Required the numbers.

Let $x + 6$, $x + 2$, $x - 2$, and $x - 6$, be the numbers;

then $(x^2 - 36) \cdot (x^2 - 4) = 1680$,

or $x^4 - 40x^2 + 144 = 1680$;

$\therefore$ by transposition, $x^4 - 40x^2 = 1536$;

completing the square, $x^4 - 40x^2 + 400 = 1936$,

extracting the root, $x^2 - 20 = \pm 44$;

$\therefore x^2 = 64$, or -24,

and $x = \pm 8$, or $\pm 2\sqrt{(-6)}$,

and $\therefore$ the numbers are ± 14, ± 10, ± 6, ± 2; the two other values of x being impossible.

15. The product of five numbers in arithmetical progression is 945, and their sum is 25. Required the numbers.

Let $x + 2y, x + y, x, x - y, x - 2y$, be the numbers;

$$\text{then } 5x = 25, \text{ and } \therefore x = 5;$$

$$\text{also, } x \cdot (x^2 - y^2) \cdot (x^2 - 4y^2) = 945, \text{ or dividing by } x = 5,$$

$$(x^2 - y^2) \cdot (x^2 - 4y^2) = 189,$$

$$\text{or } x^4 - 5x^2y^2 + 4y^4 = 189,$$

$$\text{and } 4y^4 - 125y^2 + 625 = 189;$$

$$\text{by transposition, } 4y^4 - 125y^2 = -436;$$

$$\text{completing the square, } 4y^4 - 125y^2 + \left(\frac{125}{4}\right)^2 = \frac{15625}{16} - 436 = \frac{8649}{16};$$

$$\text{extracting the root, } 2y^2 - \frac{125}{4} = \pm \frac{93}{4};$$

$$\therefore 2y^2 = \frac{109}{2}, \text{ or } 8; \ \therefore y^2 = \frac{109}{4}, \text{ or } 4;$$

$$\therefore y = \pm \frac{\sqrt{109}}{2}, \text{ or } \pm 2,$$

and the numbers are 9, 7, 5, 3, 1.

16. A Gentleman divided £210 among three servants, in geometrical progression; the first had £90 more than the last. How much had each?

Let $xy^2, xy, x =$ the number of pounds each had;

$$\text{then } xy^2 = x + 90,$$

$$\text{and } x + xy + xy^2 = 210;$$

$$\text{or } 2x + xy + 90 = 210;$$

$$\text{by transposition, } 2x + xy = 120.$$

$$\text{Now from the first equation, } x = \frac{90}{y^2 - 1},$$

$$\text{and from the last, } x = \frac{120}{y + 2};$$

$$\therefore \frac{120}{y + 2} = \frac{90}{y^2 - 1};$$

$$\text{or } \frac{4}{y + 2} = \frac{3}{y^2 - 1};$$

$$\therefore 4y^2 - 4 = 3y + 6;$$

by transposition, $4y^2 - 3y = 10$;

completing the square, $4y^2 - 3y + \frac{9}{16} = 10 + \frac{9}{16} = \frac{169}{16}$;

extracting the root, $2y - \frac{3}{4} = \pm \frac{13}{4}$,

and $2y = 4$, or $-\frac{5}{2}$;

and $\therefore y = 2$, or $-\frac{5}{4}$;

whence $x = \frac{120}{y+2} = 30$, or 160,

and the sums are 120, 60, and 30 pounds.

17. The sum of three numbers in geometrical progression is 35; and the mean term is to the difference of the extremes as 2 to 3. Required the numbers.

Let x, xy, and xy^2, be the three numbers;

$$\therefore x + xy + xy^2 = 35,$$

and $xy : xy^2 - x :: 2 : 3$,

or $y : \quad y^2 - 1 :: 2 : 3$;

$$\therefore y^2 - 1 = \frac{3}{2}.y;$$

by transposition, $y^2 - \frac{3}{2}.y = 1$;

completing the square, $y^2 - \frac{3}{2}y + \frac{9}{16} = 1 + \frac{9}{16} = \frac{25}{16}$;

extracting the root, $y - \frac{3}{4} = \pm \frac{5}{4}$;

$\therefore y = 2$, or $-\frac{1}{2}$, which last does not answer the conditions;

$$\therefore (x + 2x + 4x =) \, 7x = 35;$$

$$\therefore x = 5;$$

and the numbers are 5, 10, and 20.

18. There are three numbers in geometrical progression, the greatest of which exceeds the least by 15. Also the difference of the squares of the greatest and least is to the sum of the squares of all the three numbers as 5 : 7. Required the numbers.

Let x, xy, xy^2, be the numbers;

$$\text{then } xy^2 - x = 15,$$

$$\text{and } x^2y^4 - x^2 : x^2y^4 + x^2y^2 + x^2 :: 5 : 7,$$

$$\text{or } y^4 - 1 : y^4 + y^2 + 1 :: 5 : 7;$$

$$\therefore\ y^4 - 1 : y^2 + 2 :: 5 : 2,$$

$$\text{and } y^4 - 1 = \frac{5y^2}{2} + 5;$$

$$\text{by transposition, } y^4 - \frac{5}{2}.y^2 = 6;$$

$$\text{completing the square, } y^4 - \frac{5}{2}.y^2 + \frac{25}{16} = 6 + \frac{25}{16} = \frac{121}{16};$$

$$\text{extracting the root, } y^2 - \frac{5}{4} = \pm\frac{11}{4};$$

$$\therefore\ y^2 = 4, \text{ or } -\frac{3}{2}, \text{ which last is impossible,}$$

$$\text{and } y = \pm 2;$$

$$\therefore \text{ from the first equation, } (4x - x =)\ 3x = 15,$$

$$\text{and } x = 5;$$

$\therefore$ the numbers are 5, 10, and 20.

19. The sum of three numbers in geometrical progression is 33, and the product of the mean and the sum of the extremes is 30. Required the numbers.

Let the numbers be $\frac{x}{y}$, x, and xy;

$$\text{then } \frac{x}{y} + x + xy = 13,$$

$$\text{and } \left(\frac{x}{y} + xy\right).\ x = 30;$$

$$\therefore \text{ by transposition, } 13 - x = \frac{x}{y} + xy = \frac{30}{x};$$

$$\text{and } 13x - x^2 = 30,$$
$$\text{or } x^2 - 13x = -30,$$
$$\text{completing the square, } x^2 - 13x + \frac{169}{4} = \frac{169}{4} - 30 = \frac{49}{4};$$
$$\text{extracting the root, } x - \frac{13}{2} = \pm\frac{7}{2},$$
$$\text{and } \therefore\ x = 10, \text{ or } 3.$$
$$\text{If } x = 3;\ \text{then } \frac{3}{y} + 3y = 13 - 3 = 10,$$
$$\text{or } 3 + 3y^2 = 10y;$$
$$\text{by transposition, } 3y^2 - 10y = -3,$$
$$\text{or } y^2 - \frac{10}{3}.y = -1;$$
$$\text{completing the square, } y^2 - \frac{10}{3}.y + \frac{25}{9} = \frac{25}{9} - 1 = \frac{16}{9};$$
$$\text{extracting the root, } y - \frac{5}{3} = \pm\frac{4}{3};$$
$$\therefore\ y = 3, \text{ or } \frac{1}{3},$$

and the numbers are 1, 3, 9.

If the other value of x be taken, the corresponding values of y are impossible.

20. There are three numbers in arithmetical progression, and the square of the first added to the product of the other two is 16; the square of the second added to the product of the other two is 14. What are the numbers?

Let $x - y$, x, $x + y$, be the numbers;

$$\text{then } 2x^2 - xy + y^2 = 16,$$
$$\text{and } 2x^2 - y^2 = 14;$$
$$\therefore \text{ by subtraction, } 2y^2 - xy = 2,$$
$$\text{and by addition, } 4x^2 - xy = 30,$$
$$\text{or } 2y^2 = 2 + xy,$$
$$\text{and } 4x^2 = 30 + xy;$$
$$\therefore \text{ by multiplication, } 8x^2y^2 = 60 + 32xy + x^2y^2;$$

$$\text{by transposition, } 7x^2y^2 - 32xy = 60,$$

$$\text{or } x^2y^2 - \frac{32}{7}.xy = \frac{60}{7};$$

$$\text{completing the square, } x^2y^2 - \frac{32}{7}.xy + \frac{256}{49} = \frac{60}{7} + \frac{256}{49} = \frac{676}{49};$$

$$\text{extracting the root, } xy - \frac{16}{7} = \pm\frac{26}{7};$$

$$\therefore\ xy = 6, \text{ or } -\frac{10}{7};$$

$$\therefore\ 2y^2 = 2 + xy = 8,$$

$$\text{and } y^2 = 4;$$

$$\therefore\ y = \pm 2,$$

$$\text{and } 4x^2 = 30 + xy = 36;$$

$$\therefore\ 2x = \pm 6,$$

$$\text{and } x = \pm 3;$$

$\therefore$ the numbers are 1, 3, 5; or -5, -3, -1. The other value of xy was introduced in the operation, and does not answer the conditions of the question.

21. The sum of four whole numbers in arithmetical progression is 20, and the sum of their reciprocals is $\frac{25}{24}$. Required the numbers.

Let $x - 3y$, $x - y$, $x + y$, $x + 3y$, be the numbers;

$$\text{then } 4x = 20,$$

$$\text{or } x = 5.$$

$$\text{Again } \frac{1}{x - 3y} + \frac{1}{x - y} + \frac{1}{x + y} + \frac{1}{x + 3y} = \frac{25}{24},$$

$$\text{or } \frac{4x^3 - 20xy^2}{x^4 - 10x^2y^2 + 9y^4} = \frac{25}{24};$$

$$\therefore\ 25 \times (9y^4 - 250y^2 + 625) = 24 \times (500 - 100y^2),$$

$$\text{or } 9y^4 - 250y^2 + 625 = 24 \times (20 - 4y^2);$$

$$\text{by transposition, } 9y^4 - 154y^2 = -145,$$

$$\text{completing the square, } 9y^4 - 154y^2 + \frac{5929}{9} = \frac{5929}{9} - 145 = \frac{4624}{9};$$

$$\text{extracting the root, } 3y^2 - \frac{77}{3} = \pm\frac{68}{3},$$

$$\text{and } 3y^2 = 3, \text{ or } \frac{145}{3};$$

$$\therefore\ y^2 = 1, \text{ or } \frac{145}{9},$$

$$\text{and } y = \pm 1, \text{ or } \frac{\pm\sqrt{(145)}}{3},$$

and $\therefore$ the numbers are 2, 4, 6, 8.

22. There is a number consisting of 3 digits, the first of which is to the second as the second to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it, the digits will be inverted. Required the number.

Let the digits be represented by x, xy, xy^2;

$$\text{then } 100x + 10xy + xy^2 : x + xy + xy^2 :: 124 : 7,$$

$$\text{or } 100 + 10y + y^2 : 1 + y + y^2 :: 124 : 7;$$

$$\text{div}^{\text{do}}.\ 99 + 9y : 1 + y + y^2 :: 117 : 7,$$

$$\text{or } 11 + y : 1 + y + y^2 :: 13 : 7;$$

$$\therefore\ 13y^2 + 13y + 13 = 7y + 77;$$

$$\text{by transposition, } 13y^2 + 6y = 64,$$

$$\text{or } y^2 + \frac{6}{13}.y = \frac{64}{13};$$

$$\text{completing the square, } y^2 + \frac{6}{13}.y + \left(\frac{3}{13}\right)^2 = \frac{64}{13} + \frac{9}{(13)^2} = \frac{841}{169};$$

$$\text{extracting the root, } y + \frac{3}{13} = \pm\frac{29}{13};$$

$$\therefore\ y = 2, \text{ or } -\frac{32}{13};$$

$$\text{also } 100x + 10xy + xy^2 + 594 = 100xy^2 + 10xy + x;$$

$$\text{by transposition, } 99x + 594 = 99xy^2,$$

$$\text{or } x + 6 = xy^2 = 4x;$$

$$\therefore \text{ by transposition, } 6 = 3x,$$

$$\text{and } 2 = x;$$

$\therefore$ the digits are 2, 4, 8, and the number is 248.

23. There are five whole numbers, the three first of which are in geometric progression; the three last in arithmetic progression, the second number being the second difference. The sum of the four last = 40, and the product of the second and last = 64. Required the numbers.

Let x = the first,

and y = the common ratio of the three first;

$\therefore$ the numbers are $x, xy, xy^2, xy^2 + xy, xy^2 + 2xy$;

$$\therefore 3xy^2 + 4xy = 40,$$

$$\text{and } x^2y^3 + 2x^2y^2 = 64.$$

Multiplying the first equation by xy, and the second by 3,

$$3x^2y^3 + 4x^2y^2 = 40xy,$$

$$\text{and } 3x^2y^3 + 6x^2y^2 = 192;$$

$\therefore$ by subtraction, $2x^2y^2 = 192 - 40xy$;

by transposition, $2x^2y^2 + 40xy = 192$,

$$\text{or } x^2y^2 + 20xy = 96;$$

completing the square, $x^2y^2 + 20xy + 100 = 196$;

extracting the root, $xy + 10 = \pm 14$;

$$\therefore xy = 4, \text{ or } - 24.$$

Now from the first equation, $xy \,.\, (3y + 4) = 40$,

$$\text{or } 4 \,.\, (3y + 4) = 40;$$

$$\therefore 3y + 4 = 10;$$

by transposition, $3y = 6$,

$$\text{and } y = 2;$$

$$\therefore \text{also } x = 2,$$

and the numbers are 2, 4, 8, 12, 16.

24. There are two casks A and B; of which, A the greater holds 312 gallons. Into A a certain quantity of wine is put, and B is filled with water; then water is conveyed out of B into A in the following manner. First, a number of gallons is taken, which is less by two than the square root of the number of gallons in A, then a quantity less than the former by two gallons, and so on. Now when B is in this manner exactly emptied, A is exactly full: and it

is known that 8 gallons were taken out of B at one time, after which the quantity left in B was 12 gallons. Required the number of gallons of wine in A.

Since the quantities taken out of B are in a decreasing progression, whose common difference is 2, and one term of this progression is 8, therefore the next terms are 6, 4, 2, the sum of which is $= 12$; and therefore the quantity last drawn out is 2 gallons. Let $x^2 =$ the number of gallons of wine in A, then $x - 2$, $x - 4$, &c. are the numbers of gallons drawn each successive time; and the number of terms is evidently $\frac{x}{2} - 1$; and therefore the whole quantity drawn from B is $\frac{x}{2} \cdot \left(\frac{x}{2} - 1\right) = \frac{x^2}{4} - \frac{x}{2}$.

$$\therefore x^2 + \frac{x^2}{4} - \frac{x}{2} = 312,$$

$$\text{or } \frac{5x^2}{4} - \frac{x}{2} = 312;$$

$$\therefore x^2 - \frac{2}{5}x = \frac{1248}{5};$$

$$\text{completing the square, } x^2 - \frac{2}{5}x + \frac{1}{25} = \frac{1248}{5} + \frac{1}{25} = \frac{6241}{25};$$

$$\text{extracting the root, } x - \frac{1}{5} = \pm \frac{79}{5},$$

and $x = 16$, or $-\frac{78}{5}$, which last will not answer the conditions; therefore $x^2 = 256 =$ the number required.

25. The diagonals of 4 squares are in an increasing geometrical progression, and the product of the squares of the diagonals of the extremes is to the product of the diagonals of the means as a side of the third is to the square root of the common ratio divided by $4\sqrt{2}$. Required the diagonal of the third square, and the common ratio, supposing their difference equal to 45.

Let $\dfrac{x}{y}$, x, xy, and $xy^2 =$ the diagonals;

then since the diagonal : a side :: $\sqrt{2} : 1$;

$$\text{the side of the third} = \frac{xy}{\sqrt{2}},$$

$$\text{and } \frac{x^2}{y^2} \times x^2y^4 : x \times xy :: \frac{xy}{\sqrt{2}} : \frac{\sqrt{y}}{4\sqrt{2}};$$

$$\therefore x^2y : 1 :: 4x\sqrt{y} : 1;$$

$$\therefore x^2y = 4x\sqrt{y},$$

$$\text{and } xy^{\frac{1}{2}} = 4.$$

$$\text{Now } y - xy = 45,$$

$$\text{or } y - 4y^{\frac{1}{2}} = 45;$$

completing the square, $y - 4y^{\frac{1}{2}} + 4 = 49$;

extracting the root, $y^{\frac{1}{2}} - 2 = \pm 7$;

$\therefore$ $y^{\frac{1}{2}} = 9$, or -5, which last does not agree with the conditions; and $\therefore$ $y = 81$;

$$\text{whence } x = \frac{4}{y^{\frac{1}{2}}} = \frac{4}{9};$$

and $\therefore$ the diagonal of the third square $= 36$.

26. Two persons, *A* and *B*, traded together. *A* gained every year £3 more than the preceding year, and the last year he gained £17. His whole gain was £57. *B* in the first four years gained £52, and if what *A* put into the common stock be added to what *B* gained the second year, the sum will be £13. How many years did they remain in trade, and what were their original stocks?

Since *A*'s annual gains are in an increasing arithmetical progression, whose common difference is 3, the last term 17, and sum 57, if $n =$ the number of terms, then (*Alg.* 192),

$$\{34 - (n-1) \cdot 3\} \cdot \frac{n}{2} = 57,$$

$$\text{or } 37n - 3n^2 = 114;$$

$$\therefore n^2 - \frac{37}{3}n = -\frac{114}{3};$$

completing the square, $n^2 - \frac{37}{3}n + \left(\frac{37}{6}\right)^2 = \frac{1369}{36} - \frac{114}{3} = \frac{1}{36}$;

$$\text{extracting the root, } n - \frac{37}{6} = \pm\frac{1}{6};$$

$$\therefore n = 6, \text{ or } \frac{19}{3};$$

hence they remained 6 years in trade, and consequently *A*'s gain the first year was £2, and his gain in four years was £26.

Let $\therefore x = A$'s stock,
and $26 : 52 : x : B$'s stock $= 2x$,
and *A*'s gain the second year being £5,
$x : 2x :: 5 : B$'s gain $= 10$;
$\therefore x + 10 = 13$,
and $x = 3$;
$\therefore$ *A*'s stock was £3, and *B*'s £6.

27. A pyramidical pile of cannon-balls, the base of which was an equilateral triangle, was all used in an engagement, except the three lowest layers, and 4 balls of the next layer; these were afterwards formed into a pile with a rectangular base, having as many balls in one side of the lowest layer, as there were in the side of the lowest layer of the pyramidical pile, and 4 in the adjacent side. What was the number of balls; and what the number of layers in each pile when complete?

Let $x =$ the number of balls in a side of the lowest layer;

$$\therefore x \cdot \frac{x+1}{2} = \text{the number of balls in that layer,}$$

$$\text{and } (x-1) \cdot \frac{x}{2} = \text{the number in the next,}$$

$$\text{and } (x-2) \cdot \frac{x-1}{2} = \text{the number in the third;}$$

$$\therefore \frac{3x^2 - 3x + 2}{2} + 4 = \text{the whole number left.}$$

Now since there were only 4 balls in one side of the second pile, there can only be four layers, which will contain $4x$, $3.(x-1)$, $2.(x-2)$, and $x-3$ balls respectively;

$$\therefore \frac{3x^2-3x+2}{2}+4=10x-10,$$

$$\text{or } 3x^2-3x+2+8=20x-20;$$

$$\text{by transposition, } 3x^2-23x=-30,$$

$$\text{or } x^2-\frac{23}{3}x=-10;$$

$$\text{completing the square, } x^2-\frac{23}{3}.x+\left(\frac{23}{6}\right)^2=\frac{529}{36}-10=\frac{169}{36};$$

$$\text{extracting the root, } x-\frac{23}{6}=\pm\frac{13}{6},$$

and $x=6$, or $\frac{5}{3}$, which last cannot answer the conditions of the problem. Hence there were 6 layers in the first pile, and they contained 1, 3, 6, 10, 15, 21 balls, respectively; $\therefore$ the whole number of balls in the first pile was 56, and in the second 50.

(32.) In the preceding solutions it may be observed, that, in many instances, values of the unknown quantities are deduced, which do not agree with the conditions of the problems. This is always the case when the roots of the equations are negative; and the circumstance arises from that peculiar quality of an algebraic expression, by which it is denominated either positive or negative. The product of two or any even number of such quantities, whether all of them are positive or all negative, will only be affected with a positive sign: thus the quantity xy will represent the product of $+x\times+y$, or of $-x\times-y$; and a^2, of $+a\times+a$, or of $-a\times-a$; consequently, in the reduction of such quantities to their constituent factors by the rules of division or evolution, these factors may be considered either as all positive or all negative. But in common language, in which the conditions of a problem are expressed, quantity or number is from its very nature what in Algebra is meant by the term positive, *i.e.* it increases any

homogeneous quantity to which it is added, and diminishes any one from which it is subtracted. Hence it may be understood, why, when quadratic equations are formed to express the conditions of a problem, the resulting roots may exceed in number what appear to be required as answers to the problem, and why such as are negative cannot be applied to its conditions.

These roots or values, however, though inapplicable in their present shape, will, if assumed as positive, become correct answers to the problem under a different modification of the conditions. In the equations thence deduced, these former negative values will appear as positive roots, and the former positive values as negative roots. Thus, if Prob. 12, page 200, be transformed into the following, "A detachment from an "army was marching in regular column with 5 *fewer* in depth "than in front; but upon the enemy coming in sight the front "was increased till it became = 845 — the original front; and "by this movement the detachment was drawn up in 5 lines. "Required the number of men;" from the solution of this problem the number is found to be 3900, answering to the number which would be found from using the negative value of x in the original problem; and the equation for determining this ($x^2 - 5x = 4225 - 5x$) differs from the other only in the sign of x.

In Prob. 19, page 204, x is found to be equal to ± 4, where the negative value shows, that if the trading vessel had turned out of its first course in a direction contrary to CE, or on the opposite side of the line AC, it would have been taken after sailing 4 miles in that direction.

In Prob. 1, page 212, the negative value of x is found to be -130. But if the problem be modified so as to become "A merchant sold a quantity of brandy, by which he lost "£39 *more* than the prime cost, and found that his loss "was as much *per* cent. as the brandy cost him. What was "that price?" the equation for determining the price, is $\frac{x^2}{100} = 39 + x$, which is deduced from the equation to the

original problem by changing the sign of x, the positive value of which is in this case 130.

Also, in Prob. 4, page 213, the negative value of x is $-\frac{15}{2}$. Now if the problem were, "Bought two sorts of linen, "for the finer of which I gave 6 crowns *more than for the* "*other*. An ell of the finer cost as many shillings as there "were ells of the finer. Also 28 ells of the coarser (which "was the whole quantity) sold at such a price, that 8 ells "cost as many shillings as one ell of the finer. How many "ells were there of the finer; and what was the value of "each piece?" an equation arises differing from the equation to the original problem only in the sign of x, and whose positive root is $\frac{15}{2}$; whence there were $7\frac{1}{2}$ ells of the finer at 7*s*. 6*d*. *per* ell, the whole price of which was therefore £2. 16*s*. 3*d*., and the whole price of the coarser was £1. 6*s*. 3*d*. And in the very same manner, all the other problems may be transformed.

(33.) The same reasoning will apply to the case in which all the roots of the resulting equation are negative. None of its values can in this case be applied to satisfy the conditions of the problem; but if the conditions are properly modified, equations may be deduced, of which these values rendered positive will become roots, and will satisfy such conditions.

(34.) The same observation holds, if the resulting values be the square roots of negative quantities, with this exception, that such roots can never be applied to satisfy the conditions of the problem under any modification whatever.

SECTION XI.

Examples of the Solution of Equations, where the number of unknown Quantities exceeds the number of Equations.

(35.) It has before been observed, that when the number of unknown quantities exceeds the number of equations, some of these quantities cannot be found except in terms of the others; the values of which, being undetermined, may be assumed at pleasure; thus admitting a number of answers that will be indefinite. A problem thus not properly limited is called *an indeterminate problem.* In such, however, it is not unusual to annex the condition that the values of the numbers sought should be positive integers, or at least rational; or by other limitations to lessen the number of answers.. In the different kinds of these indeterminate problems, different expedients will be made use of; and different artifices be found appropriate to questions differently circumstanced. These, however, are best learned by practice.

(36.) In the case of a simple equation expressing the relation of two unknown quantities, whose corresponding integral values are required, the common rule is, to divide the whole equation by the less coefficient, and to assume that part of the quotient, which is in a fractional form, equal to a whole number. A new simple equation is thus obtained, in which a repetition of the process takes place. And this is similarly continued with each new equation, till the coefficient of one of the quantities becomes unity, and that of the other a whole number. An integral value of the former may then be obtained by substituting zero or any whole number for the other; and then from the preceding equations, integral values of the quantities proposed may be ascertained.

EXAMPLES.

1. Having given $2x + 3y = 35$, to find the corresponding positive integral values of x and y.

Dividing by 2 the least coefficient of the unknown quantities,

$$x = \frac{35 - 3y}{2} = 17 - y - \frac{y - 1}{2}.$$

$$\text{Assume } \frac{y-1}{2} = \text{whole number} = m,$$

$$\therefore\ y = 2m + 1;$$

$$\text{whence } x = 16 - 3m;$$

and here $3m$ must be less than 16, or m not greater than 5; the question $\therefore$ will admit of 6 answers.

Let $m = 0$,	then $x = 16$	and $y = 1$,
$m = 1$,	$x = 13$	$y = 3$,
$m = 2$,	$x = 10$	$y = 5$,
$m = 3$,	$x = 7$	$y = 7$,
$m = 4$,	$x = 4$	$y = 9$,
$m = 5$,	$x = 1$	$y = 11$.

2. Given $7x + 11y = 100$, to find the corresponding positive integral values of x and y.

Dividing by 7 the least coefficient of the unknown quantities,

$$x = 14 - y - \frac{4y - 2}{7};$$

$$\text{Assume } \frac{2y - 1}{7} = \text{whole number} = m;$$

$$\therefore\ 2y = 7m + 1,$$

$$\text{and } y = \frac{7m + 1}{2} = 3m + \frac{m+1}{2}.$$

$$\text{Again, assume } \frac{m+1}{2} = \text{whole number} = n;$$

$$\therefore\ m = 2n - 1,$$

$$\text{and } y = 7n - 3,$$

$$x = 19 - 11n;$$

where n must be less than 2, and can $\therefore$ only $= 1$; in which case $x = 8$ and $y = 4$.

3. Given $9x + 13y = 200$, to find the corresponding positive integral values of x and y.

Dividing by 9,

$$x = \frac{200 - 13y}{9} = 22 - y - \frac{4y-2}{9}.$$

Assume $\frac{2y-1}{9}$ = whole number $= m$;

$$\therefore\ 2y = 9m + 1,$$

$$\text{and } y = 4m + \frac{m+1}{2}.$$

Again, assume $\frac{m+1}{2}$ = whole number $= n$;

$$\therefore\ m = 2n - 1,$$

$$\text{and } y = 9n - 4;$$

$$\therefore\ x = 28 - 13n.$$

And since $9n$ must be greater than 4, and $13n$ less than 28; $\therefore$ n must not exceed 2, nor be less than 1.

Let $n = 1$, then $x = 15$, and $y = 5$,
$n = 2$, $x = 2$. $y = 14$.

4. Given $11x + 13y = 190$, to find corresponding positive integral values of x and y.

Dividing by 11,

$$x = \frac{190 - 13y}{11} = 17 - y - \frac{2y - 3}{11}.$$

Assume $\frac{2y-3}{11}$ = whole number $= m$;

$$\therefore\ y = 5m + 1 + \frac{m+1}{2}.$$

Again, assume $\frac{m+1}{2}$ = whole number $= n$;

$$\text{then } m = 2n - 1,$$

$$\text{and } y = 11n - 4,$$

$$x = 22 - 13n;$$

where n cannot exceed $\frac{22}{13}$, *i. e.* it cannot exceed 1, and must be greater than 0.

$$\text{Let } n = 1, \therefore x = 9, \ y = 7.$$
$$\text{If } n = 0, \ x = 22, \quad \text{and } y = -4;$$
$$\text{if } n = 2, \ x = -4, \text{ and } y = 18, \text{ \&c.}$$

5. Given $13x + 16y = 97$, find corresponding positive integral values of x and y.

Dividing by 13,

$$x = \frac{97 - 16y}{13} = 7 - y - \frac{3y - 6}{13}.$$

$$\text{Assume } \frac{y-2}{13} = \text{whole number} = m;$$

$$\therefore y = 13m + 2,$$
$$\text{and } x = 5 - 16m,$$

where it is evident that m cannot be $= 1$, to have a positive value of x.

$$\text{Let } m = 0, \text{ then } x = 5, \text{ and } y = 2.$$
$$\text{If } m = 1, \text{ then } x = -11, \text{ and } y = 15;$$

and so integral *negative* values may be obtained.

6. Given $17x + 23y = 183$, to find corresponding positive integral values of x and y.

Dividing by 17,

$$x = \frac{183 - 23y}{17} = 10 - y - \frac{6y - 13}{17}.$$

$$\text{Assume } \frac{6y - 13}{17} = \text{whole number} = m;$$

$$\therefore 6y = 17m + 13,$$

$$\text{and } y = 3m + 2 - \frac{m-1}{6}.$$

$$\text{Assume } \frac{m-1}{6} = \text{whole number} = n;$$

$$\therefore m = 6n + 1,$$
$$\text{and } y = 17n + 5,$$
$$x = 4 - 23n;$$

where $n = 0$ is the only value which will give positive integer values of x and y; viz. $x = 4$ and $y = 5$.

7. Given $19x + 5y = 119$, to find corresponding integral values of x and y.

Dividing by 5,

$$y = \frac{119 - 19x}{5} = 24 - 4x - \frac{x - 1}{5}.$$

$$\text{Assume } \frac{x-1}{5} = \text{whole number} = m;$$

$$\therefore\ x = 5m + 1,$$
$$\text{and } y = 20 - 19m;$$

where $19m$ must be less than 20, and $\therefore$ m cannot exceed 1.

$$\text{Let } m = 0,\ \therefore\ x = 1, \text{ and } y = 20,$$
$$m = 1,\ \therefore\ x = 6,\quad y = 1,$$

which are the only integral values.

8. Given $46x + 3y = 3668$, to find corresponding positive integral values of x and y.

Dividing by 3,

$$y = 1222 - 15x - \frac{x-2}{3}.$$

$$\text{Assume } \frac{x-2}{3} = \text{whole number} = m;$$

$$\therefore\ x = 3m + 2,$$
$$\text{and } y = 1192 - 46m;$$

where m must be less than $\frac{1192}{46}$, and cannot $\therefore$ exceed 25.

$$\text{Let } m = 0, \text{ then } x = 2, \text{ and } y = 1192,$$
$$m = 1,\quad x = 5,\quad y = 1146,$$
$$m = 2,\quad x = 8,\quad y = 1100,$$

and so on, by assuming m equal to every number up to 25, we obtain pairs of integral positive values of x and y, the former increasing by 3, and the latter decreasing by 46.

9. Given $3x = 8y - 16$; find the least corresponding positive integral values of x and y.

$$\text{Here } x = \frac{8y - 16}{3} = 3y - 5 - \frac{y+1}{3}.$$

$$\text{Assume } \frac{y+1}{3} = \text{whole number} = m,$$

$$\therefore\ y = 3m - 1,$$

$$\text{and } x = 8m - 8,$$

where m must be greater than 1.

Assume $m = 2$, then $x = 8$,

and $y = 5$;

the least numbers which will satisfy the conditions.

If $m = 3$, $x = 16$, $y = 8$; and other values of m will give corresponding values of x and y.

10. Given $7x - 9y = 29$; find the least corresponding positive integral values of x and y.

$$\text{Here } x = \frac{29 + 9y}{7} = 4 + y + \frac{2y+1}{7}.$$

$$\text{Assume } \frac{2y+1}{7} = \text{whole number} = m;$$

$$\therefore\ 2y = 7m - 1,$$

$$\text{and } y = 3m + \frac{m-1}{2}.$$

$$\text{Let } \frac{m-1}{2} = \text{whole number} = m;$$

$$\therefore\ m = 2n + 1,$$

$$\text{and } y = 7n + 3,$$

$$x = 9m + 8.$$

Let $m = 0$, then $x = 8$, and $y = 3$; which are the least whole numbers which satisfy the equation.

By assuming other values of m, corresponding values of x and y may be obtained.

11. Given $9x - 7y = 6$; find the least corresponding positive integral values of x and y.

$$\text{Here } y = \frac{9x - 6}{7} = x - 1 + \frac{2x+1}{7}.$$

$$\text{Assume } \frac{2x+1}{7} = \text{whole number} = m;$$

$$\therefore x = \frac{7m-1}{2} = 3m + \frac{m-1}{2}.$$

$$\text{Let } \frac{m-1}{2} = \text{whole number} = n;$$

$$\therefore m = 2n + 1;$$

$$\text{whence } x = 7n + 3,$$

$$\text{and } y = 10n + 3.$$

If $n = 0$, $x = 3$, and $y = 3$, which are the lowest integral values. Others may be obtained by assuming different values of n.

12. Given $11x - 17y = 5$; find the least corresponding integral values of x and y.

$$\text{Here } x = \frac{17y+5}{11} = 2y - \frac{5y-5}{11}.$$

$$\text{Assume } \frac{y-1}{11} = \text{whole number} = m;$$

$$\therefore y = 11m + 1,$$

$$\text{and } x = 17m + 2.$$

And if we assume $m = 0$, $x = 2$, and $y = 1$, which are the least numbers. Other values may be obtained by assuming different values of n.

13. Given $14x = 4y + 7$; determine whether positive integral values of x and y can be found.

$$\text{Here } y = \frac{14x-7}{4} = 3x - 2 + \frac{2x+1}{4}.$$

$$\text{Assume } \frac{2x+1}{4} = \text{whole number} = m;$$

$$\therefore 2x = 4m - 1;$$

which is impossible; $\therefore$ no integers can be found to answer the conditions.

14. Given $19x = 14y - 11$; find the least corresponding positive integral values of x and y.

$$\text{Here } x = \frac{19x + 11}{14} = x + \frac{5x + 11}{14}.$$

$$\text{Assume } \frac{5x + 11}{14} = \text{whole number} = m;$$

$$\therefore\ x = \frac{14m - 11}{5} = 3m - 2 - \frac{m + 1}{5}.$$

$$\text{Let } \frac{m+1}{5} = \text{whole number} = n;$$

$$\therefore\ m = 5n - 1;$$

$$\text{whence } x = 14n - 5,$$

$$\text{and } y = 19n - 6;$$

where to obtain positive integers, n cannot be $= 0$.

Let $n = 1$, then $x = 9$, and $y = 13$, the least values.

15. Given $23x - 9y = 929$; find the least corresponding positive integral values of x and y.

$$\text{Here } y = \frac{23x - 929}{9} = 2x - 103 + \frac{5x - 2}{9}.$$

$$\text{Assume } \frac{5x - 2}{9} = \text{whole number} = m;$$

$$\therefore\ x = 2m - \frac{m-2}{5}.$$

$$\text{Let } \frac{m-2}{5} = \text{whole number} = n;$$

$$\therefore\ m = 5n + 2;$$

$$\text{whence } x = 9n + 4,$$

$$\text{and } y = 23n - 93;$$

where any value of n less than 5 will give y negative.

Let $\therefore\ m = 5$, then $x = 49$, and $y = 22$, the least values.

16. Given $5x + 7y + 11z = 224$; find all the positive integral values of x, y, and z, which satisfy the equation.

$$\text{Here } x = \frac{224 - 7y - 11z}{5} = 45 - y - 2z - \frac{1 + 2y + z}{5}.$$

$$\text{Assume } \frac{1 + 2y + z}{5} = \text{whole number} = m;$$

$$\text{then } z = 5m - 2y - 1,$$

$$\text{and } x = 47 + 3y - 11m.$$

If then we assume different values for m and y, corresponding values of x and z may be obtained.

Let then $m = 1$, and $y = 1$; $\therefore x = 39$, $z = 2$; any other values of y will give $z = 0$, or negative.

Let $m = 2$, and $y = 1$,	then $x = 28$,	and $z = 7$,
$y = 2$,	$x = 31$,	$z = 5$,
$y = 3$,	$x = 34$,	$z = 3$,
$y = 4$,	$x = 37$,	$z = 1$.

Other values of y will give z negative.

It is evident that m must be greater than $\frac{2y + 1}{5}$,

and less than $\frac{47 + 3y}{11}$;

and with this limitation other values may be found.

17. Given $17x + 19y + 21z = 400$; find all the positive integral values of x, y, and z which satisfy the equation.

$$\text{Here } x = \frac{400 - 19y - 21z}{17} = 23 - y - z - \frac{2y + 4z - 9}{17}.$$

$$\text{Assume } \frac{2y + 4z - 9}{17} = \text{whole number} = m.$$

$$\therefore y = 8m - 2z + 4 + \frac{m + 1}{2}.$$

$$\text{Let } \frac{m + 1}{2} = \text{whole number} = n;$$

$$\therefore m = 2n - 1;$$

$$\text{whence } y = 17n - 4 - 2z,$$

$$\text{and } x = 28 - 19n + z.$$

Substituting for z and n different values, we obtain integral values of x and y.

T 2

$$\begin{aligned} \text{If } n = 1, \text{ and } z &= 1,\ x = 10,\ y = 11, \\ z &= 2,\ x = 11,\ y = 9, \\ z &= 3,\ x = 12,\ y = 7, \text{ \&c.} \\ z &= 6,\ x = 15,\ y = 1; \end{aligned}$$

and n cannot exceed $\frac{28 + z}{19}$, nor be less than 1.

18. Given $\left.\begin{aligned} x + 2y + 3z &= 20 \\ 4x + 5y + 6z &= 47 \end{aligned}\right\}$ find the corresponding positive integral values of x, y, and z.

From the first equation, $2x + 4y + 6z = 40$,

but $4x + 5y + 6z = 47$;

$\therefore$ by subtraction, $2x + y = 7$,

and $y = 7 - 2x$;

whence $3z = 20 - 14 + 4x - x = 6 + 3x$,

and $z = 2 + x$.

$$\begin{aligned} &\text{Let } \therefore x = 1, \text{ then } y = 5, \text{ and } z = 3, \\ &\qquad\quad x = 2, \qquad\quad y = 3, \qquad\quad z = 4, \\ &\qquad\quad x = 3, \qquad\quad y = 1, \qquad\quad z = 5; \end{aligned}$$

but if x be assumed greater than 3, y becomes negative.

19. Required the positive integral solutions of the equation $xy + 2x + 3y = 42$.

Here $(x + 3) \,.\, y = 42 - 2x$,

$$\text{and } y = \frac{42 - 2x}{x + 3} = -2 + \frac{48}{x + 3};$$

$\therefore x + 3$ is a divisor of 48.

Now the divisors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, 48.

Let then $x + 3 = m$;

$\therefore x = m - 3$,

$$\text{and } y = \frac{48}{m} - 2;$$

m $\therefore$ must be greater than 3, else x will be negative; and if less than 24, y will be negative; if $= 24$, y will $= 0$.

$$\begin{aligned}
\text{Let } m &= 4, & \text{then } x &= 1, & \text{and } y &= 10,\\
m &= 6, & x &= 3, & y &= 6,\\
m &= 8, & x &= 5, & y &= 4,\\
m &= 12, & x &= 9, & y &= 2,\\
m &= 16, & x &= 13, & y &= 1.
\end{aligned}$$

If all the integral solutions be required,

$$\begin{aligned}
\text{Let } m &= 1, & \text{then } x &= -2, & \text{and } y &= 46,\\
m &= 2, & x &= -1, & y &= 22,\\
m &= 3, & x &= 0, & y &= 14,\\
m &= 48, & x &= 45, & y &= -1.
\end{aligned}$$

20. Required the positive integral solutions of the equation $12xy = 5x + 7y + 15$.

$$\text{Here } (12x - 7) \cdot y = 5x + 15,$$

$$\text{and } y = \frac{5x + 15}{12x - 7};$$

$$\therefore\ 12y = \frac{60x + 180}{12x - 7} = 5 + \frac{215}{12x - 7};$$

and $\therefore$ $12x - 7$ must be a divisor of 215.

But the divisors of 215 are 1, 5, 43. And since if $m = 12x - 7$ $x = \frac{m+7}{12}$, such values of x only will answer as when increased by 7 will be divisible by 12, it is evident that 5 is the only one of the divisors which can be used;

$$\therefore\ x = 1,$$

$$\text{and } y = 4.$$

21. Required the integral solutions of the equation
$xy + x^2 = 2x + 3y + 29$.

$$\text{Here } (x + 3) \cdot y = 2x - x^2 + 29,$$

$$\text{and } y = \frac{2x - x^2 + 29}{x - 3} = -(x + 1) + \frac{26}{x - 3};$$

and $\therefore$ $x - 3$ must be a divisor of 26.

Let it $= m$; $\therefore$ $x = m + 3$.

Now the divisors of 26 are 1, 2, 13, 26;

$$\begin{aligned} \text{and if } m &= 1, \; x = 4, \; y = 21, \\ m &= 2, \; x = 5, \; y = 7, \\ m &= 13, \; x = 16, \; y = -15, \\ m &= 26, \; x = 29, \; y = -29. \end{aligned}$$

22. Find a number which, divided by 2 and 3, leaves remainders respectively 1 and 2.

Let $x =$ the number;

then $\frac{x-1}{2}$ is a whole number;

let it $= m$;

$$\therefore x = 2m + 1.$$

But $\frac{x-2}{3}$ is a whole number;

$$i.\,e.\ \frac{2m-1}{3} = \text{whole number} = n;$$

$$\therefore m = \frac{3n+1}{2} = n + \frac{n+1}{2}.$$

$$\text{Let } \frac{n+1}{2} = p;$$

$$\therefore n = 2p - 1,$$

$$\text{and } m = n + p = 3p - 1;$$

$$\text{whence } x = 2m + 1 = 6p - 1.$$

$$\begin{aligned} \text{Assume } p &= 1, \text{ then } x = 5, \\ p &= 2, \qquad\quad x = 11; \end{aligned}$$

and by assuming different values of p, other numbers are obtained which answer the conditions.

23. Find the least whole number which, when divided by 3 and 5, has its respective remainders 1 and 3.

Let $x =$ the number;

then $\frac{x-1}{3} =$ whole number $= m$;

$$\therefore x = 3m + 1.$$

$$\text{But } \frac{x-3}{5} = \text{whole number};$$

$$\therefore \frac{3m-2}{5} = \text{whole number} = n.$$

$$\text{and } m = \frac{5n+2}{3} = n + \frac{2n+2}{3}.$$

$$\text{Let } \frac{n+1}{3} = \text{whole number} = p;$$

$$\therefore n = 3p - 1,$$

$$\text{and } m = n + 2p = 5p - 1;$$

$$\therefore x = 3m + 1 = 15p - 2.$$

Let $p = 1$, then $x = 13$; the least whole number which will answer the conditions.

24. Required a number which, divided by 11, leaves a remainder 3, but being divided by 19 leaves a remainder 5.

$$\text{Let } x = \text{the number};$$

$$\text{then } \frac{x-3}{11} = \text{whole number} = m;$$

$$\therefore x = 11m + 3.$$

$$\text{But } \frac{x-5}{19} = \frac{11m-2}{19} = \text{whole number} = n;$$

$$\therefore \frac{19n+2}{11} = n + \frac{8n+2}{11}.$$

$$\text{Let } \frac{4n+1}{11} = p;$$

$$\therefore 4n = 11p - 1,$$

$$\text{and } n = \frac{11p-1}{4} = 3p - \frac{p+1}{4}.$$

$$\text{Assume } \frac{p+1}{4} = r;$$

$$\therefore p = 4r - 1,$$

$$\text{and } n = 11r - 3,$$

$$m = 19r - 5;$$

$$\text{whence } x = 209r - 52.$$

Let $r = 1$, then $x = 157$, the least number which can answer the conditions. Other values may be obtained by assuming $r = 1, 2, 3$, &c.

25. Find the least whole number which, divided by 19, leaves a remainder 7, and divided by 28 will leave a remainder 13.

Let $x =$ the number;

$$\text{then } \frac{x-7}{19} = \text{whole number} = m,$$

$$\text{and } x = 19m + 7.$$

$$\text{Also } \frac{x-13}{28} = \frac{19m-6}{28} = \text{whole number} = n;$$

$$\therefore\ 19m = 28n + 6,$$

$$\text{and } m = n + \frac{9n+6}{19}.$$

$$\text{Let } \frac{3n+2}{19} = \text{whole number} = p;$$

$$\text{then } n = \frac{19p-2}{3} = 6p + \frac{p-2}{3}.$$

$$\text{Assume } \frac{p-2}{3} = \text{whole number} = r;$$

$$\text{then } p = 3r + 2,$$

$$n = 19r + 12,$$

$$m = 28r + 18,$$

$$\text{and } x = 532r + 349.$$

Assume $r = 0$; then $x = 349$, the least number that will answer the conditions. Other numbers, however, may be determined by assuming different values of r.

26. A certain number, when divided by 5 and 4, leaves a remainder 1; but when divided by 3, leaves no remainder. Determine the number.

Let $x =$ the number;

$$\text{then } \frac{x-1}{5} = \text{whole number} = m;$$

$$\therefore\ x = 5m + 1.$$

$$\text{Also } \frac{x-1}{4} = \frac{5m}{4} = m + \frac{m}{4} = \text{whole number.}$$

$$\text{Let } \frac{m}{4} = \text{whole number} = n;$$

$$\therefore\ m = 4n,$$

$$\text{and } x = 20n + 1.$$

$$\text{But } \frac{x}{3} = \text{whole number} = \frac{20n+1}{3} = 7n - \frac{n-1}{3}.$$

$$\text{Assume } \frac{n-1}{3} = p;$$

$$\text{then } n = 3p + 1,$$

$$m = 12p + 4,$$

$$\text{and } x = 60p + 21.$$

If then p be assumed $= 0$, $x = 21$, the least whole number.

If $p = 1$, $x = 81$;

and so on for every number which may be assumed for p.

27. Find a number which, divided by 3, 4, 5, respectively, shall leave 2, 3, 4 for remainders.

$$\text{Let } x = \text{the number};$$

$$\text{then } \frac{x-2}{3} = \text{whole number} = m;$$

$$\therefore\ x = 3m + 2.$$

$$\text{Also } \frac{x-3}{4} = \frac{3m-1}{4} = \text{whole number} = n;$$

$$\therefore\ m = \frac{4n+1}{3} = n + \frac{n+1}{3},$$

$$\text{and } \frac{n+1}{3} = \text{whole number} = p;$$

$$\therefore\ n = 3p - 1,$$

$$\text{and } m = 4p - 1,$$

$$\text{whence } x = 12p - 1.$$

$$\text{But } \frac{x-4}{5} = \frac{12p-5}{5} = \text{whole number} = 2p - 1 + \frac{2p}{5},$$

$$\therefore\ \frac{2p}{5} = \text{whole number, and } \frac{p}{5} = \text{whole number} = r;$$

$$\therefore p = 5r,$$
$$\text{and } x = 12p - 1 = 60r - 1.$$
$$\text{Assume } r = 1, \text{ then } x = 59;$$
$$r = 2, \qquad x = 119;$$
$$r = 3, \qquad x = 179, \text{ \&c.}$$

28. Find a number less than 400 which is a multiple of 7, and upon being divided by 2, 3, 4, 5, 6, always leaves 1 for a remainder.

$$\text{Let } x = \text{the number};$$
$$\text{then } \frac{x-1}{2} = \text{whole number} = m;$$
$$\therefore x = 2m + 1,$$
$$\text{and } \frac{x-1}{3} = \text{whole number} = \frac{2m}{3}; \text{ let it} = n;$$
$$\therefore m = \frac{3n}{2} = n + \frac{n}{2}.$$
$$\text{Assume } \frac{n}{2} = p;$$
$$\therefore n = 2p,$$
$$m = 3p,$$
$$\text{and } x = 6p + 1.$$
$$\text{Also } \frac{x-1}{4} = \frac{6p}{4} = \frac{3p}{2} = \text{whole number};$$
$$\therefore \frac{p}{2} = \text{whole number} = q,$$
$$\text{and } p = 2q;$$
$$\therefore x = 12q + 1.$$
$$\text{Again } \frac{x-1}{5} = \frac{12q}{5} = \text{whole number} = 2q + \frac{2q}{5}.$$
$$\text{Let } \frac{q}{5} = r;$$
$$\therefore q = 5r,$$
$$\text{and } x = 60r + 1.$$
$$\text{Also } \frac{x-1}{6} = \frac{60r}{6} = 10r = \text{whole number}.$$

By assuming $\therefore r = 1, 2, 3, 4, 5$, the values of x will be found to be 61, 121, 181, 241, 301, 361, which are all less than 400. But 301 is the only multiple of 7; the only number $\therefore$ which answers the conditions.

29. Divide 25 into two parts, one of which may be divisible by 2, and the other by 3.

$$\text{Let } 2x = \text{one part,}$$
$$\text{and } 3y = \text{the other;}$$
$$\text{then } 2x + 3y = 25,$$
$$\text{and } x = \frac{25 - 3y}{2} = 12 - y - \frac{y-1}{2}.$$
$$\text{Assume } \frac{y-1}{2} = \text{whole number} = m;$$
$$\therefore y = 2m + 1,$$
$$\text{and } x = 11 - 3m.$$

And since $11 - 3m$ is a positive number, $3m$ must be less than 11, and $\therefore m$ less than 4.

Let $m = 0$, then $y = 1$, and $x = 11$, $\therefore$ the parts are 22 and 3;
$m = 1$, $y = 3$, $x = 8$, $\therefore$ the parts are 16 and 9;
$m = 2$, $y = 5$, $x = 5$, $\therefore$ the parts are 10 and 15;
$m = 3$, $y = 7$, $x = 2$, $\therefore$ the parts are 4 and 21.

And these are the only divisions which can be made.

30. How many ways are there of paying £7 with crowns and seven-shilling pieces?

Let x = the number of crowns,
and y = the number of seven-shilling pieces, } required to pay the sum.

$$\text{then } 5x + 7y = 140,$$
$$\text{and } x = 28 - y - \frac{2y}{5}.$$
$$\text{Let } \frac{y}{5} = m;$$
$$\therefore y = 5m,$$
$$\text{and } x = 28 - 7m.$$

And as m cannot be $= \frac{28}{7}$, or 4, there can be only 3 answers, or 3 different ways of payment.

Suppose $m = 1$, then $x = 21$, and $y = 5$;
$m = 2$, $x = 14$, $y = 10$;
$m = 3$, $x = 7$, $y = 15$.

31. In how many ways may £80 be paid in sovereigns and guineas?

Let x = the number of sovereigns, y = the number of guineas, $\}$ required to pay the sum.

then $20x + 21y = 80 \times 20 = 1600$,

$$\text{and } x = 80 - y - \frac{y}{20}.$$

$$\text{But } \frac{y}{20} = \text{whole number} = m;$$

$$\therefore y = 20m,$$
$$\text{and } x = 80 - 21m.$$

And since $80 - 21m$ must be a positive whole number, $\therefore$ m must be less than $\frac{80}{21}$, and cannot $\therefore$ exceed 3. There are $\therefore$ only three ways of payment.

Suppose $m = 1$, then $x = 59$, and $y = 20$;
$m = 2$, $x = 38$, $y = 40$;
$m = 3$, $x = 17$, $y = 60$.

32. Can £100 be paid in guineas and moidores?

If it can, let x and y be the numbers of each.

then $21x + 27y = 2000$,

$$\text{and } x = \frac{2000 - 27y}{21} = 95 - y - \frac{6y - 5}{21}.$$

$$\text{But } \frac{6y - 5}{21} = \text{whole number} = m;$$

$$\therefore y = 3m + \frac{3m + 5}{6}.$$

$$\text{Assume } \frac{3m + 5}{6} = n;$$

$$\therefore m = \frac{6n - 5}{3} = 2n - \frac{5}{3},$$

which is impossible; and $\therefore$ the payment cannot be made.

33. In how many different ways is it possible to exchange 11 bullocks which are worth £12 each, for sheep which are worth £2. 5*s.* each, and pigs which are worth 12*s.* each?

Let x and y represent the numbers of sheep and pigs which must be given in exchange;

$$\text{then } 45x + 12y = 11 \times 240,$$

$$\text{or } 15x + 4y = 11 \times 80 = 880;$$

$$\therefore y = 220 - 4x + \frac{x}{4};$$

$$\text{Assume } \frac{x}{4} = m;$$

$$\therefore x = 4m,$$

$$\text{and } y = 220 - 15m;$$

where it is evident that m cannot exceed 14, nor be less than 1; there are $\therefore$ 14 different ways of exchange.

$$\text{Let } m = 1, \text{ then } x = 4, \text{ and } y = 205;$$

$$m = 2, \quad x = 8, \quad y = 190;$$

and so on, the values of x increasing by 4, and those of y decreasing by 15.

34. A company of men and women club for the payment of a reckoning; each man pays 25*s.*, and each woman 16*s.*; and it is found that all the women together pay one shilling more than the men. How many men and women were there?

$$\text{Let } x = \text{the number of men,}$$

$$\text{and } y = \text{the number of women;}$$

$$\text{then } 16y = 25x + 1,$$

$$\text{and } y = x + \frac{9x + 1}{16}.$$

$$\text{Let } \frac{9x + 1}{16} = \text{whole number} = m;$$

$$\therefore 9x = 16m - 1,$$

$$\text{and } x = 2m - \frac{2m + 1}{9}.$$

$$\text{Assume } \frac{2m + 1}{9} = \text{whole number} = n;$$

$$\text{then } 2m = 9n - 1,$$

$$\text{and } m = 4n + \frac{n-1}{2}.$$

$$\text{Let } \frac{n-1}{2} = p;$$

$$\text{then } n = 2p + 1,$$

$$m = 9p + 4;$$

$$\text{whence } x = 16p + 7,$$

$$\text{and } y = 25p + 11.$$

Let $p = 0$, then $x = 7$, and $y = 11$; the least numbers which answer the conditions. Other values of x and y may be obtained by assuming different values for p; the number of men increasing by 16, and the women by 25.

35. A person distributes 4*s.* 2*d.* among some beggars, giving 7*d.* to some, and a shilling each to the rest. How many were there?

Let x = the number of those to whom 7*d.* was given,
y = the number to whom 1*s.* was given;

$$\text{then } 7x + 12y = 50,$$

$$\text{and } x = \frac{50 - 12y}{7} = 7 - y - \frac{5y - 1}{7}.$$

$$\text{But } \frac{5y-1}{7} = \text{whole number} = m;$$

$$\therefore\ 5y = 7m + 1,$$

$$\text{and } y = m + \frac{2m+1}{5}.$$

$$\text{And } \frac{2m+1}{5} = \text{whole number} = n;$$

$$\therefore\ m = 2n + \frac{n-1}{2}.$$

$$\text{But } \frac{n-1}{2} = \text{whole number} = p;$$

$$\therefore\ n = 2p + 1,$$

$$\text{and } m = 5p + 2;$$

$$\text{whence } y = m + n = 7p + 3,$$

$$\text{and } x = 7 - y - m = 2 - 12p.$$

And as $12p$ must be less than 2, no whole number substituted for it will answer the conditions: but if $p = 0$, $x = 2$, and $y = 3$, the numbers required.

36. *A* wishes to pay a debt of £1. 12*s*., but has only half-crowns in his pocket, while *B* has only fourpenny pieces. How may they settle the matter most simply between them?

Suppose *A* to pay x half-crowns, and receive y fourpenny pieces;

$$\text{then } 30x - 4y = 32 \times 12,$$
$$\text{or } 15x - 2y = 192;$$
$$\therefore y = 7x - 96 + \frac{x}{2}.$$
$$\text{Let } \frac{x}{2} = m;$$
$$\therefore x = 2m,$$
$$\text{and } y = 15m - 96;$$

whence m must be greater than 6.

Let $m = 7$; $\therefore x = 14$, and $y = 9$; the smallest number of coins which will answer the conditions.

37. It is required to divide 24 into three such parts that if the first be multiplied by 36, the second by 24, and the third by 8, the sum of these products may be 516.

Let x, y, and z be the three parts;

$$\text{then } x + y + z = 24,$$
$$\text{and } 36x + 24y + 8z = 516,$$
$$\text{or } 9x + 6y + 2z = 129;$$
$$\text{but } 2x + 2y + 2z = 48;$$
$$\therefore \text{by subtraction } 7x + 4y = 81,$$
$$\text{and } y = 20 - 2x + \frac{x+1}{4}.$$
$$\text{Let } \frac{x+1}{4} = \text{whole number} = m;$$
$$\therefore x = 4m - 1,$$
$$\text{and } y = 22 - 7m,$$
$$z = 3 + 3m;$$

where it is evident that m cannot be so great as $\frac{22}{7}$, and $\therefore$ cannot exceed 3.

$$\begin{aligned} \text{Let } m &= 1, \text{ then } x = 3, \quad y = 15, \text{ and } z = 6, \\ m &= 2, \qquad\quad x = 7, \quad y = 8, \qquad\quad z = 9, \\ m &= 3, \qquad\quad x = 11, \ \ y = 1, \qquad\quad z = 12. \end{aligned}$$

38. In how many different ways is it possible to pay £10 in crowns, seven-shillings, and moidores?

If there be x crowns, y seven-shillings, and z moidores,

$$5x + 7y + 27z = 200,$$

$$\text{and } x = \frac{200 - 7y - 27z}{5} = 40 - y - 5z - \frac{2y + 2z}{5}.$$

$$\text{Let } \frac{y + z}{5} = m;$$

$$\therefore\ y = 5m - z,$$

$$\text{and } x = 40 - 7m - 4z;$$

where m cannot be equal to $\frac{36}{7}$, and $\therefore$ cannot exceed 5.

Suppose $m = 1$, then $x = 33 - 4z$,
and $y = 5 - z$;

where z cannot $= 5$.

$$\begin{aligned} \text{If } z &= 1, \quad x = 29, \quad y = 4, \\ z &= 2, \quad x = 25, \quad y = 3, \\ z &= 3, \quad x = 21, \quad y = 2, \\ z &= 4, \quad x = 17, \quad y = 1. \end{aligned}$$

Suppose now $m = 2$, then $x = 26 - 4z$,
and $y = 10 - z$;

where z cannot exceed 6.

$$\begin{aligned} \text{If } z &= 1, \quad x = 22, \quad y = 9, \\ z &= 2, \quad x = 18, \quad y = 8, \\ z &= 3, \quad x = 14, \quad y = 7, \\ z &= 4, \quad x = 10, \quad y = 6, \\ z &= 5, \quad x = 6, \quad y = 5, \\ z &= 6, \quad x = 2, \quad y = 4. \end{aligned}$$

Again, suppose $m = 3$, then $x = 19 - 4z$,

$$y = 15 - z;$$

where z cannot exceed 4.

$$\text{If } z = 1,\quad x = 15,\quad y = 14,$$
$$z = 2,\quad x = 11,\quad y = 13,$$
$$z = 3,\quad x = 7,\quad y = 12,$$
$$z = 4,\quad x = 3,\quad y = 11.$$

Suppose $m = 4$, then $x = 12 - 4z$,

$$y = 20 - z;$$

where z cannot exceed 2.

$$\text{Let } z = 1,\quad x = 8,\quad y = 19,$$
$$z = 2,\quad x = 4,\quad y = 18.$$

Suppose $m = 5$, then $x = 5 - 4z$,

and $y = 25 - z$;

here z cannot exceed 1.

Let $z = 1$, then $x = 1$,

$$y = 24.$$

39. Forty-one persons, men, women, and children, spent 40*s*., whereof each man paid 4*s*., each woman 3*s*., and each child 4*d*. How many of each were there?

Let x, y, and z be the numbers respectively,

$$\text{then } x + y + z = 41,$$
$$\text{and } 4x + 3y + \tfrac{1}{3}z = 40;$$
$$\text{whence } 12x + 9y + z = 120,$$
$$\text{but } x + y + z = 41;$$
$$\therefore \text{ by subtraction, } 11x + 8y = 79,$$
$$\text{and } y = \frac{79 - 11x}{8} = 9 - x - \frac{3x - 7}{8}.$$
$$\text{Let } \frac{3x - 7}{8} = m;$$
$$\therefore x = \frac{8m + 7}{3} = 3m + 2 - \frac{m - 1}{3}.$$
$$\text{Let } \frac{m - 1}{3} = n;$$

$$\therefore\ m = 3n + 1,$$
$$\text{and } x = 8n + 5,$$
$$y = 3 - 11n,$$
$$\text{and } z = 33 + 3n.$$

Here it is evident that n must $= 0$, and

$$\therefore\ x = 5,\ y = 3,\ z = 33;$$

the only numbers which satisfy the conditions.

40. Find two square numbers whose sum is a square.

Let x^2 and y^2 = the two squares,

then $x^2 + y^2$ is a square; let it $= (mx - y)^2$,

$$x^2 + y^2 = m^2x^2 - 2mxy + y^2;$$
$$\therefore\ (m^2 - 1) \cdot x^2 = 2mxy,$$
$$\text{and } x = \frac{2my}{m^2 - 1},$$
$$\text{and } x^2 + y^2 = \left\{\frac{4m^2}{(m^2-1)^2} + 1\right\} \cdot y^2 = \left\{\frac{m^2+1}{m^2-1}\right\}^2 \cdot y^2;$$

which is a square whatever be the values of m and y.

By assuming $\therefore$ different values for them, different numbers may be found which answer the conditions; thus, if $m = 2$, $x = \frac{4y}{3}$, and $x^2 + y^2 = \frac{25y^2}{9}$ a square; and taking different values of y, as 1, 2, 3, &c., the sum of the squares will be found to $= \frac{25}{9}, \frac{100}{9}, 25$, &c., which are squares.

If $m = 3$, $x = \frac{6y}{8} = \frac{3y}{4}$, and $x^2 + y^2 = \frac{25y^2}{16}$ a square, and assuming different values of y, the sum of the squares will be found to $= \frac{25}{16}, \frac{100}{16}, \frac{225}{16}$, &c.

If $m = 4$, $x = \frac{8y}{15}$, and $x^2 + y^2 = \frac{289}{225}y^2$ a square, and taking different values of y, the sum of the squares will be found to be $\frac{289}{225}, \frac{1156}{225}, \frac{2601}{225}$, &c.

But if the numbers be required to be integers, suppose $y = m^2 - 1$, then $x = 2m$, and the sum of their squares will be $(m^2 + 1)^2$.

$$\text{Let } m = 2, \text{ then } x = 4, \text{ and } y = 3,$$
$$\text{and } x^2 + y^2 = 16 + 9 = 25 = (5)^2.$$
$$\text{Let } m = 3, \text{ then } x = 6, \text{ and } y = 8,$$
$$\text{and } x^2 + y^2 = 36 + 64 = 100 = (10)^2.$$
$$\text{If } m = 4, \text{ then } x = 8, \text{ and } y = 15,$$
$$\text{and } x^2 + y^2 = 64 + 225 = 289 = (17)^2,$$

and by assuming $m = 5$, 6, &c., other integer values of x and y may be obtained.

41. Find two square numbers whose sum shall be equal to a given square (a^2).

$$\text{Let } x^2 \text{ and } y^2 = \text{the numbers},$$
$$\text{then } x^2 + y^2 = a^2,$$
$$\text{and } y = (a^2 - x^2)^{\frac{1}{2}} = a - mx, \text{ suppose};$$
$$\text{then } a^2 - x^2 = a^2 - 2max + m^2x^2,$$
$$\text{and } 2max = (m^2 + 1) \,.\, x^2;$$
$$\therefore\ x = \frac{2ma}{m^2 + 1},$$
$$\text{and } x^2 = \frac{4m^2a^2}{(m^2 + 1)^2};$$
$$\therefore\ y^2 = a^2 - x^2 = a^2 - \frac{4m^2a^2}{(m^2 + 1)^2} = \left(\frac{m^2 - 1}{m^2 + 1}\right)^2 . \, a^2,$$

whatever be the value of m.

Let it be assumed $= 2$, 3, 4, &c., the corresponding values of x^2 and y^2 will be

$$\frac{16a^2}{25}, \quad \frac{9a^2}{25},$$
$$\frac{36a^2}{100}, \quad \frac{64a^2}{100},$$
$$\frac{64a^2}{289}, \quad \frac{225a^2}{289}, \text{ \&c.}$$

If $(m^2 + 1)^2$ be assumed $= a^2$, then $x^2 = 4m^2$,

and $y^2 = (m^2 - 1)^2$;

and the sum of their squares is $= (m^2 + 1)^2 = a^2$.

Let $m = 2$, $a^2 = 25$, and $x^2 = 16$, $y^2 = 9$,

$m = 3$, $a^2 = 100$, $x^2 = 36$, $y^2 = 64$, &c.

42. Find a number such that if 2 be subtracted from the double of its square, the remainder may be a square.

Let $x =$ the number,

then $2x^2 - 2$ is a square, $= 2 \,.\, (x + 1) \,.\, (x - 1)$.

Assume it $= m^2 \,.\, (x + 1)^2$;

$$\therefore\ 2 \,.\, (x + 1) \,.\, (x - 1) = m^2 \,.\, (x + 1)^2,$$

and $2x - 2 = m^2x + m^2$,

or $(2 - m^2) \,.\, x = m^2 + 2$;

$$\therefore\ x = \frac{2 + m^2}{2 - m^2}.$$

Let $m = 1$, then $x = 3$, and $2x^2 - 2 = 16 = (4)^2$,

$m = \frac{3}{2}$, $x = -17$, $2x^2 - 2 = 576 = (24)^2$.

43. Find a number whose square doubled and increased by 2 may likewise be a square.

Let $x =$ the number,

then $2x^2 + 2$ is a square.

This is $= 4 + 2x^2 - 2 = 4 + 2 \,.\, (x + 1) \,.\, (x - 1)$.

Assume this $= \{2 + m \,.\, (x + 1)\}^2$,

then $4 + 2 \,.\, (x + 1) \,.\, (x - 1) = 4 + 4m \,.\, (x + 1) + m^2 \,.\, (x + 1)^2$,

and $2 \,.\, (x - 1) = 4m + m^2 \,.\, (x + 1)$;

$$\therefore\ (2 - m^2) \,.\, x = m^2 + 4m + 2,$$

$$\text{and } x = \frac{m^2 + 4m + 2}{2 - m^2}.$$

If $m = 0$, $x = 1$, and $2x^2 + 2 = 4$,

$m = 1$, $x = 7$, $2x^2 + 2 = 100$.

44. Find a value of x which will make $\frac{x^2 + x}{2}$ a rational square.

$$\text{Here } \frac{x^2 + x}{2} = y^2,$$

$$\text{and } 4x^2 + 4x + 1 = 8y^2 + 1, \text{ which is a square.}$$

$$\text{Assume it} = (1 + my)^2;$$

$$\therefore 8y^2 + 1 = 1 + 2my + m^2y^2,$$

$$\text{and } (8 - m^2) \,.\, y^2 = 2my;$$

$$\therefore y = \frac{2m}{8 - m^2};$$

$$\text{whence } 2x + 1 = (8y^2 + 1)^{\frac{1}{2}} = \frac{8 + m^2}{8 - m^2};$$

$$\text{whence } 2x = \frac{8 + m^2}{8 - m^2} - 1 = \frac{2m^2}{8 - m^2},$$

$$\text{and } x = \frac{m^2}{8 - m^2}.$$

$$\text{Hence } \frac{x^2 + x}{2} = \frac{1}{2}\left\{\frac{m^4}{(8 - m^2)^2} + \frac{m^2}{8 - m^2}\right\} = \frac{4m^2}{(8 - m^2)^2};$$

which is a square whatever be the value of m.

If $m = 1$, $x = \frac{1}{7}$, and the function $= \frac{4}{49}$;

if $m = 2$, $x = 1$, and the function $= 1$;

if $m = \frac{3}{2}$, $x = \frac{9}{23}$, and the function $= \frac{144}{529}$;

if $m = \frac{8}{3}$, $x = 8$, and the function $= 36$.

45. Find two square numbers whose sum and product shall be equal.

$$\text{Let } x^2 \text{ and } y^2 = \text{the two numbers};$$

$$\text{then } x^2 + y^2 = x^2y^2,$$

$$\text{and } x^2 = \frac{y^2}{y^2 + 1};$$

which, being a square, $y^2 + 1$ must evidently be a square.

And this will be the case when $y = \frac{m^2 + n^2}{2mn}$;

$$\text{for then } x^2 = \frac{y^2}{y^2 - 1} = \frac{(m^2 + n^2)^2}{(m^2 - n^2)^2},$$

$$\text{and } y^2 = \frac{(m^2 + n^2)^2}{4m^2n^2}.$$

$$\text{Let } m = 2, \quad n = 1, \quad \text{then } x^2 = \frac{25}{9}, \quad y^2 = \frac{25}{16},$$

$$m = 3, \quad n = 2, \quad \text{then } x^2 = \frac{169}{25}, \quad y^2 = \frac{169}{144}, \text{ \&c.}$$

46. Determine three numbers in Arithmetical Progression such that the sum of every two of them, diminished by the third, shall be a square number.

$$\text{Let } \left.\begin{array}{l} x^2 - y \\ x^2 \\ x^2 + y \end{array}\right\} \text{ be the numbers;}$$

then $x^2 - 2y$, $x^2 + 2y$, and x^2 must be squares.

$$\text{Let } x^2 - 2y = (m - x)^2 = m^2 - 2mx + x^2,$$

$$\text{and } x^2 + 2y = (n - x)^2 = n^2 - 2nx + x^2;$$

$$\text{then } 2y = 2mx - m^2,$$

$$\text{and } 2y = n^2 - 2nx;$$

$$\text{whence } 2mx - m^2 = n^2 - 2nx,$$

$$\text{and } 2 \,.\, (m + n) \,.\, x = m^2 + n^2;$$

$$\therefore\ x = \frac{m^2 + n^2}{2 \,.\, (m + n)}.$$

$$\text{And } 2y = 2m \,.\, \frac{m^2 + n^2}{2 \,.\, (m + n)} - m^2 = m \,.\, \frac{m^2 + n^2 - m^2 - mn}{m + n};$$

$$\therefore\ y = \frac{m}{2} \,.\, \frac{n^2 - mn}{m + n} = \frac{mn}{2} \,.\, \frac{n - m}{n + m};$$

where m and n may be assumed at pleasure.

$$\text{Let } m = 1, \ n = 2; \ \therefore\ x = \frac{5}{6}, \text{ and } y = \frac{1}{3} = \frac{2}{6};$$

$$\text{whence } x^2 - y = \frac{13}{36}, \quad x^2 = \frac{25}{36}, \quad x^2 + y = \frac{37}{36};$$

and to have the numbers integral, the denominators may be rejected; in which case the required numbers will be 13, 25, and 37.

SECTION XII.

PRAXIS.

I. *Simple Equations involving only one unknown Quantity.*

1. GIVEN $19x + 13 = 59 - 4x$, to find the value of x.

   ANSWER $x = 2$.

2. Given $3x + 4 - \frac{x}{3} = 46 - 2x$, to find the value of x.

   ANS. $x = 9$.

3. Given $x^2 + 15x = 35x - 3x^2$, to find the value of x.

   ANS. $x = 5$.

4. Given $\frac{x}{6} - \frac{x}{4} + 10 = \frac{x}{3} - \frac{x}{2} + 11$, to find the value of x.

   ANS. $x = 12$.

5. Given $\frac{x + 1}{5} + 3 = \frac{2x - 3}{3}$, to find the value of x.

   ANS. $x = 9$.

6. Given $\frac{7x + 2}{3} + 5x = 28 + \frac{5x - 6}{7}$, to find the value of x.

   ANS. $x = 4$.

7. Given $\frac{3x + 4}{5} + 2x = \frac{22 - x}{5} + 16$, to find the value of x.

   ANS. $x = 7$.

8. Given $\frac{7-x}{2} + 4 = \frac{3x-11}{4} + \frac{8x+15}{6}$, to find the value of x.

Ans. $x = 3$.

9. Given $\frac{2x-5}{18} + \frac{19-x}{3} = \frac{10x-7}{9} - \frac{5}{2}$, to find the value of x.

Ans. $x = 7$.

10. Given $x - \frac{2x+1}{3} = \frac{x+3}{4}$, to find the value of x.

Ans. $x = 13$.

11. Given $\frac{3x+5}{8} - \frac{21+x}{3} = 39 - 5x$, to find the value of x.

Ans. $x = 9$.

12. Given $4x - \frac{19+2x}{5} = 15 - \frac{7x+11}{4}$, to find the value of x.

Ans. $x = 3$.

13. Given $\frac{21-3x}{3} - \frac{4x+6}{9} = 6 - \frac{5x+1}{4}$, to find the value of x.

Ans. $x = 3$.

14. Given $7\frac{1}{8} + \frac{3x-1}{4} - \frac{7x+3}{16} = \frac{8x+19}{8}$, to find the value of x.

Ans. $x = 7$.

15. Given $\frac{6x+8}{11} - \frac{5x+3}{2} = \frac{27-4x}{3} - \frac{3x+9}{2}$, to find the value of x.

Ans. $x = 6$.

16. Given $x + \frac{27-9x}{4} - \frac{5x+2}{6} = \frac{61}{12} - \frac{2x+5}{3} - \frac{29+4x}{12}$, to find the value of x.

Ans. $x = 5$.

17. Given $\frac{7x-8}{11} + \frac{15x+8}{13} = 3x - \frac{31-x}{2}$, to find the value of x.

Ans. $x = 9$.

18. Given $\frac{5x-1}{2} - \frac{7x-2}{10} = 6\frac{3}{5} - \frac{x}{2}$, to find the value of x.

Ans. $x = 3$.

19. Given $\frac{3x-3}{4} - \frac{3x-4}{3} = 5\frac{1}{3} - \frac{27+4x}{9}$, to find the value of x.

Ans. $x = 9$.

20. Given $\frac{4x-34}{17} - \frac{258-5x}{3} = \frac{69-x}{2}$, to find the value of x.

Ans. $x = 51$.

21. Given $2x - \frac{4x-2}{13} = \frac{2x+11}{5} - \frac{7-8x}{7}$, to find the value of x.

Ans. $x = 7$.

22. Given $\frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}$, to find the value of x.

Ans. $x = 72$.

23. Given $\frac{7x+9}{8} - \frac{3x+1}{7} = \frac{9x-13}{4} - \frac{249-9x}{14}$, to find the value of x.

Ans. $x = 9$.

24. Given $5 - \frac{3x+25}{4} - \frac{17-6x}{9} = 2\frac{1}{36} + x - \frac{9x+40}{8}$, to find the value of x.

Ans. $x = 4$.

25. Given $\frac{7x-43}{12}+13\frac{1}{2}-\frac{5+4x}{6}=256-\frac{3x-12}{9}$ $-\frac{5x+29}{8}-12x$, to find the value of x.

Ans. $x=19$.

26. Given $4x+\frac{1}{10}-\frac{3x-13}{16}-\frac{12+7x}{9}=7x-33-\frac{9+5x}{10}$ $-\frac{11x-17}{8}$, to find the value of x.

Ans. $x=15$.

27. Given $\frac{31+4x}{3}-\frac{3x+47}{8}-\frac{3x-19}{16}=47\frac{3}{4}+\frac{16-10x}{11}$ $-\frac{5x+20}{7}$, to find the value of x.

Ans. $x=17$.

28. Given $\frac{3a+x}{x}-5=\frac{6}{x}$, to find the value of x.

Ans. $x=\frac{3a-6}{4}$.

29. Given $\frac{5}{6}.ab+\frac{4}{5}ac-\frac{2}{3}.cx=\frac{3}{4}.ac+2ab-6cx$, to find the value of x.

Ans. $x=\frac{70ab-3ac}{320c}$.

30. Given $\frac{11x-13}{25}+\frac{19x+3}{7}-\frac{5x-25\frac{1}{2}}{4}=28\frac{1}{4}-\frac{17x+4}{21}$, to find the value of x.

Ans. $x=8$.

31. Given $\frac{7x-13\frac{1}{2}}{11}-\frac{2}{3}.\frac{x-15}{7}=\frac{15}{14}.(x-1)$, to find the value of x.

Ans. $x=2\frac{4}{5}$.

32. Given $\frac{3x-2}{4}+\frac{x}{2}-11\frac{5}{6}=\frac{x-\frac{4x-9}{3}}{6}-5$, to find the value of x.

 ANS. $x=6$.

33. Given $\frac{\frac{3x}{2}-4}{6}-\frac{4x-7}{9}+x=\frac{8-\frac{x+4}{2}}{3}+2$, to find the value of x.

 ANS. $x=4$.

34. Given $\frac{1}{2}\cdot\left\{\frac{2}{3}x+4\right\}-\frac{7\frac{1}{2}-x}{3}=\frac{x}{2}\cdot\left\{\frac{6}{x}-1\right\}$, to find the value of x.

 ANS. $x=3$.

35. Given $\frac{x-1\frac{25}{36}}{2}-\frac{2-6x}{13}=x-\frac{5x-\frac{10-3x}{4}}{39}$, to find the value of x.

 ANS. $x=11$.

36. Given $\frac{4x-17}{9}-\frac{3\frac{2}{3}-22x}{33}=x-\frac{6}{x}\cdot\left\{1-\frac{x^2}{54}\right\}$, to find the value of x.

 ANS. $x=3$.

37. Given $\frac{x^2+1}{4x^2-1}=\frac{x}{1+2x}-\frac{1}{4}$, to find the value of x.

 ANS. $x=-\frac{3}{4}$.

38. Given $\frac{a^2x}{bc}-\frac{d^2}{a}+bx=\frac{ex}{f}-b+(d+b)\,.\,x$, to find the value of x.

 ANS. $x=\frac{(d^2-ab)\,.\,bcf}{a^2f-abce-abcdf}$.

39. Given $\frac{ax}{b} + \frac{cx}{d} + \frac{ex}{f} - g = h$, to find the value of x.

Ans. $x = \frac{(g + h) \,.\, bdf}{adf + bcf + bde}$.

40. Given $\frac{a^2 x}{b - c} - dc = bx - ac$, to find the value of x.

Ans. $x = \frac{c \,.\, (b - c) \,.\, (d - a)}{a^2 - b^2 + bc}$.

41. Given $\frac{2x}{3} - \frac{1 - \frac{x}{2}}{4x} = \frac{x - 1}{2} + \frac{x}{6} + \frac{7}{12}$, to find the value of x.

Ans. $x = 6$.

42. Given $\frac{9x + 20}{36} = \frac{4x - 12}{5x - 4} + \frac{x}{4}$, to find the value of x.

Ans. $x = 8$.

43. Given $\frac{20x + 36}{25} + \frac{5x + 20}{9x - 16} = \frac{4x}{5} + \frac{86}{25}$, to find the value of x.

Ans. $x = 4$.

44. Given $\frac{10x + 17}{18} - \frac{12x + 2}{13x - 16} = \frac{5x - 4}{9}$, to find the value of x.

Ans. $x = 4$.

45. Given $\frac{18x - 19}{28} + \frac{11x + 21}{6x + 14} = \frac{9x + 15}{14}$, to find the value of x.

Ans. $x = 7$.

46. Given $\frac{2x + 8\frac{1}{2}}{9} - \frac{13x - 2}{17x - 32} + \frac{x}{3} = \frac{7x}{12} - \frac{x + 16}{36}$, to find the value of x.

Ans. $x = 4$.

47. Given $\frac{7x + 6}{28} - \frac{2x + 4\frac{2}{7}}{23x - 6} + \frac{x}{4} = \frac{11x}{21} - \frac{x - 3}{42}$, to find the value of x.

Ans. $x = 4$.

48. Given $\frac{6 - 5x}{15} - \frac{7 - 2x^2}{14 \cdot (x - 1)} = \frac{1 + 3x}{21} - \frac{2x - \frac{11}{5}}{6} + \frac{1}{105}$, to find the value of x.

Ans. $x = 4$.

49. Given $\frac{x}{2} - \frac{\frac{2x - 3}{3} - \frac{3x - 1}{4}}{\frac{x - 1}{2}} = \frac{3}{2} \cdot \frac{x^2 + 2}{3x - 2}$, to find the value of x.

Ans. $x = \frac{13}{3}$.

50. Given $\frac{a \cdot (b^2 + x^2)}{bx} = ac + \frac{ax}{b}$, to find the value of x.

Ans. $x = \frac{b}{c}$.

51. Given $\frac{cx^m}{a + bx} = \frac{dx^m}{e + fx}$, to find the value of x.

Ans. $x = \frac{ad - ce}{cf - bd}$.

52. Given $\frac{a}{bx} + \frac{c}{dx} + \frac{e}{fx} + \frac{g}{hx} = k$, to find the value of x.

Ans. $x = \frac{adfh + bcfh + bdeh + bdfg}{bdfhk}$.

53. Given $(a + x) \cdot (b + x) - a \cdot (b + c) = \frac{a^2 c}{b} + x^2$, to find the value of x.

Ans. $x = \frac{ac}{b}$.

54. Given $\frac{10 + x}{5} : \frac{4x - 9}{7} :: 14 : 5$, to find the value of x.

Ans. $x = 4$.

55. Given $\frac{17 - 4x}{4} : \frac{15 + 2x}{3} - 2x :: 5 : 4$, to find the value of x.

Ans. $x = 3$.

56. Given $16x + 5 : \frac{4x + 14}{9x + 31} :: 36x + 10 : 1$, to find the value of x.

Ans. $x = 5$.

57. Given $\frac{4x + 3}{6x - 43} : 1 :: 2x + 19 : 3x - 19$, to find the value of x.

Ans. $x = 8$.

58. Given $5x + \frac{7x + 9}{4x + 3} = 9 + \frac{10x^2 - 18}{2x + 3}$, to find the value of x.

Ans. $x = 3$.

59. Given $\sqrt[3]{(10x + 35)} - 1 = 4$, to find the value of x.

Ans. $x = 9$.

60. Given $\sqrt[5]{(9x - 4)} + 6 = 8$, to find the value of x.

Ans. $x = 4$.

61. Given $\sqrt{(x + 16)} = 2 + \sqrt{x}$, to find the value of x.

Ans. $x = 9$.

62. Given $\sqrt{(x - 32)} = 16 - \sqrt{x}$, to find the value of x.

Ans. $x = 81$.

63. Given $\sqrt{(4x + 21)} = 2\sqrt{x} + 1$, to find the value of x.

Ans. $x = 25$.

64. Given $a\sqrt[3]{(bx - c)} = d \,.\, \sqrt[3]{(ex + fx - g)}$, to find the value of x.

$$\text{Ans. } x = \frac{a^3 c - d^3 g}{a^3 b - d^3 \,.\, (e + f)}.$$

65. Given $\sqrt[3]{(a^2 + c)} = \sqrt[4]{\left(\frac{a^2 + c}{d \,.\, (x + b)}\right)}$, to find the value of x.

$$\text{Ans. } x = \frac{1}{d\sqrt[3]{(a^2 + c)}} - b.$$

66. Given $\sqrt[m]{(a + x)} = \sqrt[2m]{x^2 + 5ax + b^2}$, to find the value of x.

$$\text{Ans. } x = \frac{a^2 - b^2}{3a}.$$

67. Given $a + b \,.\, \sqrt[m]{(x + d)} = c$, to find the value of x.

$$\text{Ans. } x = \left(\frac{c - a}{b}\right)^m - d.$$

68. Given $\frac{\sqrt{9x} - 4}{\sqrt{x} + 2} = \frac{15 + \sqrt{9x}}{\sqrt{x} + 40}$, to find the value of x.

Ans. $x = 4$.

69. Given $\frac{\sqrt{x} + \sqrt{b}}{\sqrt{x} - \sqrt{b}} = \frac{a}{b}$, to find the value of x.

$$\text{Ans. } x = b \,.\, \left(\frac{a + b}{a - b}\right)^2.$$

70. Given $\frac{\sqrt{6x} - 2}{\sqrt{6x} + 2} = \frac{4\sqrt{6x} - 9}{4\sqrt{6x} + 6}$, to find the value of x.

Ans. $x = 6$.

71. Given $\frac{5x - 9}{\sqrt{5x} + 3} - 1 = \frac{\sqrt{5x} - 3}{2}$, to find the value of x.

Ans. $x = 5$.

72. Given $\frac{1}{3} - \frac{7x-1}{6\frac{1}{2}-3x} = \frac{8}{3} \cdot \frac{x-\frac{1}{2}}{x-2}$, to find the value of x.

Ans. $x = \frac{14}{13}$.

73. Given $\frac{3x}{2} - \frac{81x^2-9}{(3x-1)\cdot(x+3)} = 3x - \frac{3}{2} \cdot \frac{2x^2-1}{x+3} - \frac{57-3x}{2}$, to find the value of x.

Ans. $x = 10$.

74. Given $\sqrt{\{1 + x\sqrt{(x^2+12)}\}} = 1 + x$, to find the value of x.

Ans. $x = 2$.

75. Given $\frac{ax}{b} \cdot \sqrt{(c^2x^2 + d^2)} + \frac{acx^2}{b} = ex$, to find the value of x.

Ans. $x = \frac{b^2e^2 - a^2d^2}{2abce}$.

76. Given $\sqrt{x} + \sqrt{(x-9)} = \frac{36}{\sqrt{(x-9)}}$, to find the value of x.

Ans. $x = 25$.

77. Given $\frac{2}{19}\{\sqrt{(x^2+39x+374)} - \sqrt{(x^2+20x+51)}\}$ $= \sqrt{\left(\frac{x+22}{x+17}\right)}$, to find the value of x.

Ans. $x = 78$.

II. *Simple Equations involving two unknown Quantities.*

1. Given $\left.\begin{aligned} x + 15y &= 53, \\ \text{and } y + 3x &= 27, \end{aligned}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 8, \\ y = 3. \end{cases}$

2. Given $\left.\begin{array}{l} 4x + 9y = 51, \\ \text{and } 8x - 13y = 9, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 6, \\ y = 3. \end{cases}$$

3. Given $\left.\begin{array}{l} \frac{x}{6} + \frac{y}{4} = 6, \\ \text{and } \frac{x}{4} + \frac{y}{6} = 5\frac{1}{3}, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 12, \\ y = 16. \end{cases}$$

4. Given $\left.\begin{array}{l} \frac{x}{8} + 8y = 194, \\ \text{and } \frac{y}{8} + 8x = 131, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 16, \\ y = 24. \end{cases}$$

5. Given $\left.\begin{array}{l} \frac{3x - 1}{5} + 3y - 4 = 15, \\ \text{and } \frac{3y - 5}{6} + 2x - 8 = 7\frac{1}{3}, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 7, \\ y = 5. \end{cases}$$

6. Given $\left.\begin{array}{l} 9x + \frac{8y}{5} = 70, \\ \text{and } 7y - \frac{13x}{3} = 44, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 6, \\ y = 10. \end{cases}$$

7. Given $\dfrac{7+x}{5} - \dfrac{2x-y}{4} = 3y - 5,$ and $\dfrac{5y-7}{2} + \dfrac{4x-3}{6} = 18 - 5x,$ to find the values of x and y.

Ans. $\begin{cases} x = 3, \\ y = 2. \end{cases}$

8. Given $x + 1 - \dfrac{3y+4x}{7} = 7 - \dfrac{9y+33}{14},$ and $y - 3 - \dfrac{5x-4y}{2} = x - \dfrac{11y-19}{4},$ to find the values of x and y.

Ans. $\begin{cases} x = 6, \\ y = 5. \end{cases}$

9. Given $4x + \dfrac{15-x}{4} = 2y + 5 + \dfrac{7x+11}{16},$ and $3y - \dfrac{2x+y}{5} = 2x + \dfrac{2y+4}{3},$ to find the values of x and y.

Ans. $\begin{cases} x = 3, \\ y = 4. \end{cases}$

10. Given $x - \dfrac{3x+5y}{17} + 17 = 5y + \dfrac{4x+7}{3},$ and $\dfrac{22-6y}{3} - \dfrac{5x-7}{11} = \dfrac{x+1}{6} - \dfrac{8y+5}{18},$ to find the values of x and y.

Ans. $\begin{cases} x = 8, \\ y = 2. \end{cases}$

11. Given $\dfrac{7x-21}{6} + \dfrac{3y-x}{3} = 4 + \dfrac{3x-19}{2},$ and $\dfrac{2x+y}{2} - \dfrac{9x-7}{8} = \dfrac{3y+9}{4} - \dfrac{4x+5y}{16},$ to find the values of x and y.

Ans. $\begin{cases} x = 9, \\ y = 4. \end{cases}$

12. Given $\dfrac{a}{b+y} = \dfrac{b}{3a+x}$, and $ax + 2by = c$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = \dfrac{2b^2 - 6a^2 + c}{3a}, \\ y = \dfrac{3a^2 - b^2 + c}{3b}. \end{cases}$$

13. Given $\dfrac{7x+6}{11} + \dfrac{4y-9}{3} = 3x - \dfrac{13-x}{2} - \dfrac{3y-x}{5}$, and $3x + 4 : 2y - 3 :: 5 : 3$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 7, \\ y = 9. \end{cases}$$

14. Given $\dfrac{5x+13}{2} - \dfrac{8y-3x-5}{6} = 9 + \dfrac{7x-3y+1}{3}$, and $\dfrac{x+7}{3} : \dfrac{3y-8}{4} + 4x :: 4 : 21$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 5, \\ y = 4. \end{cases}$$

15. Given $x + y : 4x + y :: 4 : 7$, and $\dfrac{\frac{11y}{6} - 2x}{5} - \dfrac{21-3y}{4} = \dfrac{2 + \frac{x}{10}}{3} - \dfrac{1}{12}$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 2, \\ y = 6. \end{cases}$$

16. Given $\dfrac{3x+4y+3}{10} - \dfrac{2x+7-y}{15} = 5 + \dfrac{y-8}{5}$, and $\dfrac{9y+5x-8}{12} - \dfrac{x+y}{4} = \dfrac{7x+6}{11}$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 7, \\ y = 9. \end{cases}$$

17. Given $13x + \frac{4y - 17 + x}{12} - \frac{15 - 3x}{4} = \frac{12y + 11}{3} - \frac{12x + 7y + 28}{6}$,

and $\frac{9x + 18}{4} - \frac{12 + 5y - 6x}{5} = \frac{15x - 3y - 5}{8} - \frac{7x + y - 10}{15}$

to find the values of x and y.

Ans. $\begin{cases} x = 2, \\ y = 11. \end{cases}$

18. Given $3x + 5y = \frac{(8a - 2b) \cdot ab}{a^2 - b^2}$,

and $a^2x - \frac{acb^2}{a + b} + (a + b + c) \cdot by = b^2x + (a + 2b)\,ab$,

to find the values of x and y.

Ans. $\begin{cases} x = \dfrac{ab}{a - b}, \\ y = \dfrac{ab}{a + b}. \end{cases}$

19. Given $\frac{2x + y}{9} + \frac{7y + 6x + 11}{18} = 9\frac{1}{2} - \frac{5x - 17}{6}$,

and $\frac{5x + 3y + 2}{7} : \frac{9y + 6}{2} :: 1 : 3$,

to find the values of x and y.

Ans. $\begin{cases} x = 7, \\ y = 4. \end{cases}$

20. Given $\frac{3x - 5y}{3} - \frac{2x - 8y - 9}{12} = \frac{y}{2} + \frac{1}{3} + \frac{1}{4}$,

and $\frac{x}{7} + \frac{y}{4} + 1\frac{1}{3} : 4x - \frac{y}{8} - 24 :: 3\frac{1}{3} : 3\frac{1}{2}$,

to find the values of x and y.

Ans. $\begin{cases} x = 7, \\ y = 4. \end{cases}$

21. Given $(x+5)\cdot(y+7) = (x+1)\cdot(y-9) + 112,$ and $2x + 10 = 3y + 1,$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 3, \\ y = 5. \end{cases}$$

22. Given $\dfrac{6x+9}{4} + \dfrac{3x+5y}{4x-6} = 3\tfrac{1}{4} + \dfrac{3x+4}{2},$ and $\dfrac{8y+7}{10} + \dfrac{6x-3y}{2y-8} = 4 + \dfrac{4y-9}{5},$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 7, \\ y = 9. \end{cases}$$

23. Given $4x - 34\tfrac{1}{3} - \dfrac{4y+13x}{27-6y} = \dfrac{12x+8}{3},$ and $3x + \dfrac{21-4y}{4x-10} = \dfrac{18x+13}{6} - 2\tfrac{1}{9},$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 7, \\ y = 5. \end{cases}$$

24. Given $16x + 6y - 1 = \dfrac{128x^2 - 18y^2 + 217}{8x - 3y + 2},$ and $\dfrac{10x + 10y - 35}{2x + 2y + 3} = 5 - \dfrac{54}{3x + 2y - 1},$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 6, \\ y = 5. \end{cases}$$

25. Given $4x + 3y + \dfrac{24 + \dfrac{11y}{2}}{2x+1} = \dfrac{16x^2 + 12xy - 8x + 5y + 28}{4x - 2},$ and $2x + 4 = 3y + \dfrac{8x^2 - 18y^2 + 108}{4x + 6y + 3},$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 3, \\ y = 2. \end{cases}$$

26. Given $3x + 6y + 1 = \dfrac{6x^2 + 130 - 24y^2}{2x - 4y + 3}$, and $3x - \dfrac{151 - 16x}{4y - 1} = \dfrac{9xy - 110}{3y - 4}$, to find the values of x and y.

Ans. $\begin{cases} x = 9, \\ y = 2. \end{cases}$

27. Given $3y + 11 = \dfrac{4x^2 - y \cdot (x + 3y)}{x - y + 4} + 31 - 4x$, and $(x + 7) \cdot (y - 2) + 3 = 2xy - (y - 1) \cdot (x + 1)$, to find the values of x and y.

Ans. $\begin{cases} x = 4, \\ y = 3. \end{cases}$

28. Given $\dfrac{2}{3}\left\{x - \dfrac{3}{5}y\right\} + \dfrac{x + \dfrac{y}{5}}{6} = \dfrac{1}{3} - \dfrac{1}{2}\left\{\dfrac{\dfrac{4}{5}y - 2}{6} - (x - y)\right\}$, and $x - 2y - \dfrac{3y - 5x}{2} = \dfrac{11}{2}(x + y) - 3 \cdot (x - y)$, to find the values of x and y.

Ans. $\begin{cases} x = \dfrac{10}{7}, \\ y = \dfrac{5}{42}. \end{cases}$

29. Given $\dfrac{x - 6}{7y} + \dfrac{4x + 7}{24} - \dfrac{x - \dfrac{y}{7}}{6} = \dfrac{19 + y}{42} - \dfrac{\dfrac{11x}{3} + 6}{56y}$, and $12x - 15y + \dfrac{13}{4} : 10y - 8x + \dfrac{86}{3} :: 93 - 9x : 6x - \dfrac{14}{5}$, to find the values of x and y.

Ans. $\begin{cases} x = 9, \\ y = 7. \end{cases}$

30. Given $\dfrac{\dfrac{7x}{4}+6y}{5}-\dfrac{\dfrac{3y+6}{5}-\dfrac{3x-2}{10}}{8}=5-\dfrac{x}{16},$ and $\dfrac{3x}{2}+\dfrac{2y}{3}+2\frac{1}{2}:\dfrac{x}{2}-\dfrac{y}{3}+\dfrac{1}{6}::10\frac{1}{2}:1\frac{1}{6},$ to find the values of x and y.

Ans. $\begin{cases} x=4, \\ y=3. \end{cases}$

31. Given $\dfrac{4x-2y+3}{3}-\dfrac{18-x+5y}{7}=\dfrac{x}{4}-\dfrac{y}{5}-\dfrac{1}{7}-7\frac{7}{10},$ and $2x-y+15:y-2x+15::\dfrac{x}{3}-\dfrac{y}{4}+\dfrac{3}{4}:\dfrac{y}{4}-\dfrac{x}{3}+\dfrac{1}{12},$ to find the values of x and y.

Ans. $\begin{cases} x=18, \\ y=24. \end{cases}$

32. Given $4+\dfrac{12y-\dfrac{6y+2}{x}}{11}=y+\dfrac{\dfrac{3xy-31}{11}+10x+13}{3x},$ and $\dfrac{2x}{3}-\dfrac{3x-5}{y+7}=\dfrac{4xy+\dfrac{170}{3}}{6y+27},$ to find the values of x and y.

Ans. $\begin{cases} x=7, \\ y=2. \end{cases}$

33. Given $\sqrt{y}-\sqrt{(y-x)}=\sqrt{(20-x)},$ and $\sqrt{(y-x)}:\sqrt{(20-x)}::3:2,$ to find the values of x and y.

Ans. $\begin{cases} x=16, \\ y=25. \end{cases}$

34. Given $\left.\begin{aligned} x + y &= a, \\ x + z &= b, \\ y + z &= c, \end{aligned}\right\}$ to find the values of x, y, and z.

$$\text{Ans.}\ \begin{cases} x = \frac{1}{2} \cdot (a + b - c), \\ y = \frac{1}{2} \cdot (a - b + c), \\ z = \frac{1}{2} \cdot (c - a + b). \end{cases}$$

35. Given $\left.\begin{aligned} x - y - z &= 6, \\ 3y - x - z &= 12, \\ 7z - y - x &= 24, \end{aligned}\right\}$ to find the values of x, y, and z.

$$\text{Ans.}\ \begin{cases} x = 39, \\ y = 21, \\ z = 12. \end{cases}$$

36. Given $\left.\begin{aligned} x + y - z &= 8, \\ 2x - y + 3z &= 21, \\ 4z + 3y - 2x &= 17, \end{aligned}\right\}$ to find the values of x, y, and z.

$$\text{Ans.}\ \begin{cases} x = 7, \\ y = 5, \\ z = 4. \end{cases}$$

37. Given $\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{5}{6}, \\ \frac{1}{x} + \frac{1}{z} &= \frac{3}{4}, \\ \frac{1}{y} + \frac{1}{z} &= \frac{7}{12}, \end{aligned}\right\}$ to find the values of x, y, and z.

$$\text{Ans.}\ \begin{cases} x = 2, \\ y = 3, \\ z = 4. \end{cases}$$

38. Given $\left.\begin{array}{r} x + 2y + 3z = 17, \\ y + 2z + 3x = 13, \\ z + 2x + 3y = 12, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = 1, \\ y = 2, \\ z = 4. \end{cases}$

39. Given $\left.\begin{array}{r} x - y + z = 30, \\ 8x - 4y + 2z = 50, \\ 27x - 9y + 3z = 64, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = \frac{2}{3}, \\ y = 7, \\ z = 36\frac{1}{3}. \end{cases}$

40. Given $\left.\begin{array}{r} 3x - y + z = 15, \\ 5x + 3y - 2z = 16, \\ 7x + 4y - 5z = 11, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = 4, \\ y = 2, \\ z = 5. \end{cases}$

41. Given $\left.\begin{array}{r} 2x + 4y - 3z = 22, \\ 4x - 2y + 5z = 18, \\ 6x + 7y - z = 63, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = 3, \\ y = 7, \\ z = 4. \end{cases}$

42. Given $\left.\begin{array}{r} 3x + 2y - z = 20, \\ 2x + 3y + 6z = 70, \\ x - y + 6z = 41, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = 5, \\ y = 6, \\ z = 7. \end{cases}$

43. Given $\left.\begin{aligned} 7x + 12y + 4z &= 128, \\ 3x + 3y + 7z &= 60, \\ 6x + y + 5z &= 68, \end{aligned}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = 8, \\ y = 5, \\ z = 3. \end{cases}$

44. Given $\left.\begin{aligned} 6x + 3y - 4z &= 22, \\ 4x - y + 6z &= 20, \\ 5x + 2y - 6z &= 11, \end{aligned}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = 3, \\ y = 4, \\ z = 2. \end{cases}$

45. Given $\left.\begin{aligned} 11x - 10y &= \frac{12y - 11z}{3}, \\ \frac{x + z - 2y}{3} &= \frac{z - y - 1}{2}, \\ 3x &= y + z + 7, \end{aligned}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = 10, \\ y = 11, \\ z = 12. \end{cases}$

46. Given $\left.\begin{aligned} \frac{x}{6} - \frac{y}{5} + \frac{z}{4} &= 11, \\ \frac{x}{5} + \frac{y}{4} - \frac{z}{3} &= 35, \\ \frac{x}{4} - \frac{y}{3} + \frac{z}{2} &= 16, \end{aligned}\right\}$ to find the values of x, y, and z.

Ans. $\begin{cases} x = 120, \\ y = 60, \\ z = 12. \end{cases}$

III. *Pure Quadratics and others which may be solved without completing the Square.*

1. Given $3x^2 - 4 = 28 + x^2$, to find the values of x.

   Ans. $x = \pm 4$.

2. Given $\left.\begin{array}{r} x + y : y :: 3 : 1, \\ \text{and } xy = 18, \end{array}\right\}$ to find the values of x and y.

   Ans. $\begin{cases} x = \pm 6, \\ y = \pm 3. \end{cases}$

3. Given $\left.\begin{array}{r} x - y : y :: 4 : 5, \\ \text{and } x^2 + 4y^2 = 181, \end{array}\right\}$ to find the values of x and y.

   Ans. $\begin{cases} x = \pm 9, \\ y = \pm 5. \end{cases}$

4. Given $\left.\begin{array}{r} x + y : x - y :: a : b, \\ \text{and } xy = c^2, \end{array}\right\}$ to find the values of x and y.

   Ans. $\begin{cases} x = \pm c \sqrt{\left(\dfrac{a+b}{a-b}\right)}, \\ y = \pm c \sqrt{\left(\dfrac{a-b}{a+b}\right)}. \end{cases}$

5. Given $\left.\begin{array}{r} x^2 + y^2 : x^2 - y^2 :: 17 : 8, \\ \text{and } xy^2 = 45, \end{array}\right\}$ to find the values of x and y.

   Ans. $\begin{cases} x = 5, \\ y = 3. \end{cases}$

6. Given $\left.\begin{array}{r} x^2 - xy = 54, \\ \text{and } xy - y^2 = 18, \end{array}\right\}$ to find the values of x and y.

   Ans. $\begin{cases} x = \pm 9, \\ y = \pm 3. \end{cases}$

7. Given $\left.\begin{array}{r} x + y : x^2 - y^2 :: 1 : 4, \\ \text{and } xy = 21, \end{array}\right\}$ to find the values of x and y.

   Ans. $\begin{cases} x = 7, \text{ or } -3, \\ y = 3, \text{ or } -7. \end{cases}$

8. Given $\left.\begin{array}{l} ax^2 + bxy = c^2, \\ \text{and } x - y : x :: m : n, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \pm c \cdot \sqrt{\left(\dfrac{nc}{na + nb - mb}\right)}, \\ y = \pm \dfrac{n-m}{n} \cdot c \cdot \sqrt{\left(\dfrac{nc}{na + nb - mb}\right)}. \end{cases}$$

9. Given $\left.\begin{array}{r} x^3 + y^3 : x^3 - y^3 :: 559 : 127, \\ \text{and } x^2y = 294, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 7, \\ y = 6. \end{cases}$$

10. Given $\left.\begin{array}{r} x^2 - xy : xy - y^2 :: 3 : 7, \\ \text{and } xy^2 = 147, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 3, \\ y = 7. \end{cases}$$

11. Given $\left.\begin{array}{r} \sqrt{x} + \sqrt{y} : \sqrt{x} - \sqrt{y} :: 4 : 1, \\ \text{and } x - y = 16, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 25, \\ y = 9. \end{cases}$$

12. Given $\left.\begin{array}{r} \sqrt[4]{x} - \sqrt[4]{y} = 3, \\ \text{and } \sqrt[4]{x} + \sqrt[4]{y} = 7, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 625, \\ y = 16. \end{cases}$$

13. Given $\left.\begin{array}{r} x - y : \sqrt{x} - \sqrt{y} :: 8 : 1, \\ \text{and } \sqrt{xy} = 15, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 25, \text{ or } 9, \\ y = 9, \text{ or } 25. \end{cases}$$

14. Given $\left.\begin{array}{r} x^3 - y^3 : x^2y - xy^2 :: 7 : 2, \\ \text{and } x + y = 6, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 4, \text{ or } 2, \\ y = 2, \text{ or } 4. \end{cases}$$

15. Given $\left.\begin{array}{l}\frac{1}{x}+\frac{1}{y}=\frac{1}{2}, \\ \text{and } \frac{2}{xy}=\frac{1}{9},\end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 6, \text{ or } 3, \\ y = 3, \text{ or } 6. \end{cases}$

16. Given $\left.\begin{array}{l}x^4 - y^4 = 369, \\ \text{and } x^2 - y^2 = 9,\end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 5, \\ y = \pm 4. \end{cases}$

17. Given $\left.\begin{array}{l}x^3 - y^3 = 56, \\ \text{and } x - y = \frac{16}{xy},\end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 4, \text{ or } -2, \\ y = 2, \text{ or } -4. \end{cases}$

18. Given $\frac{1}{1-\sqrt{(1-x^2)}} - \frac{1}{1+\sqrt{(1-x^2)}} = \frac{\sqrt{3}}{x^2}$, to find the values of x.

Ans. $x = \pm \frac{1}{2}$.

19. Given $\left.\begin{array}{l}x^2y + y^2 = 116, \\ \text{and } xy^{\frac{1}{2}} + y = 14,\end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 5, \text{ or } 2\sqrt{\frac{2}{5}}, \\ y = 4, \text{ or } 10. \end{cases}$

20. Given $\left.\begin{array}{l}\sqrt[3]{x} + \sqrt[3]{y} = 6, \\ \text{and } x + y = 72,\end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 64, \text{ or } 8, \\ y = 8, \text{ or } 64. \end{cases}$

21. Given $4x^2 + \frac{5}{2} = \frac{x^2}{y} + 10y,$ and $x^2 + 3y = 55,$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 5, \\ y = 10. \end{cases}$

22. Given $\frac{1}{x + \sqrt{(2 - x^2)}} + \frac{1}{x - \sqrt{(2 - x^2)}} = ax,$ to find the values of x.

Ans. $x = \pm \sqrt{\left(\frac{a+1}{a}\right)}.$

23. Given $\frac{x}{\sqrt{(a^2 + x^2)} - x} = b,$ to find the values of x.

Ans. $x = \pm \frac{ab}{\sqrt{(2b + 1)}}.$

24. Given $\sqrt{(\frac{1}{2}x + 2)} - \sqrt{(\frac{1}{2}x - 2)} = \sqrt{(x + 3)} - \sqrt{(x - 3)},$ to find the values of x.

Ans. $x = \pm 5.$

25. Given $\frac{1}{2}\sqrt{(4y - x)} + \frac{1}{2}\sqrt{(y - x)} = \sqrt{(2y - x)},$ and $\frac{5}{4}\sqrt{(x^2 - 6y)} + \sqrt{(y^2 - 9x)} : \sqrt{(x^2 - 6y)} :: 1\frac{3}{4} : 1,$ to find the values of x and y.

Ans. $\begin{cases} x = 7, \\ y = 8. \end{cases}$

26. Given $\frac{9}{8} \cdot \frac{\sqrt[3]{(x + y)}}{y} + \frac{9}{8} \cdot \frac{\sqrt[3]{(x + y)}}{x} = 1\frac{1}{7},$ and $\frac{7}{4} \cdot \frac{\sqrt[3]{(x - y)}}{y} - \frac{7}{4} \cdot \frac{\sqrt[3]{(x - y)}}{x} = \frac{1}{9},$ to find the values of x and y.

Ans. $\begin{cases} x = \frac{9}{2}, \\ y = \frac{7}{2}. \end{cases}$

27. Given $\left.\begin{array}{l} x^{\frac{4}{3}} + y^{\frac{2}{5}} = 20, \\ \text{and } x^{\frac{2}{3}} + y^{\frac{1}{5}} = 6, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \pm 8, \text{ or } \pm \sqrt{8}, \\ y = 32, \text{ or } 1024. \end{cases}$$

28. Given $\left.\begin{array}{l} x^4 + 2x^2y^2 + y^4 = 1296 - 4xy \,.\, (x^2 + xy + y^2), \\ \text{and } x - y = 4, \end{array}\right\}$

to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 5, \text{ or } -1, \\ y = 1, \text{ or } -5. \end{cases}$$

29. Given $(a^{4b} + 1) \,.\, (x^{\frac{1}{2}} - 1)^2 = 2 \,.\, (x + 1)$, to find the values of x.

$$\text{Ans. } x = \left(\frac{a^{2b} \pm 1}{a^{2b} \mp 1}\right)^2.$$

30. Given $\dfrac{\sqrt{a} - \sqrt{(a - x)}}{\sqrt{a} + \sqrt{(a - x)}} = a$, to find the value of x.

$$\text{Ans. } x = \frac{4a^2}{(a + 1)^2}.$$

31. Given $\left.\begin{array}{l} \dfrac{\sqrt{x} + \sqrt{(x - y)}}{\sqrt{x} - \sqrt{(x - y)}} = 4, \\ \text{and } \sqrt{x} : \sqrt{y} :: \sqrt{y} : 4, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \dfrac{625}{16}, \\ y = 25. \end{cases}$$

32. Given $\left.\begin{array}{l} \dfrac{1}{y} - \dfrac{1}{x} = \dfrac{1}{4}, \\ \text{and } x^2y - xy^2 = 16, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 4, \text{ or } -2, \\ y = 2, \text{ or } -4. \end{cases}$$

33. Given $\left.\dfrac{\sqrt{(4x+1)}+\sqrt{4x}}{\sqrt{(4x+1)}-\sqrt{4x}}=9,\right\}$ to find the value of x.

$$\text{Ans. } x=\frac{4}{9}.$$

34. Given $\left.\dfrac{a+x+\sqrt{(2ax+x^2)}}{a+x-\sqrt{(2ax+x^2)}}=b,\right\}$ to find the values of x.

$$\text{Ans. } x=\pm a\,.\,\frac{(\sqrt{b}\mp 1)^2}{2\sqrt{b}}.$$

35. Given $\left(\dfrac{2x+3}{2x-3}\right)^{\frac{1}{3}}+\left(\dfrac{2x-3}{2x+3}\right)^{\frac{1}{3}}=\dfrac{8}{13}\cdot\dfrac{4x^2+9}{4x^2-9}$, to find the values of x.

$$\text{Ans. } x=\frac{27}{14}.$$

36. Given $\left.\begin{array}{l}\sqrt{(x-y)}+\frac{1}{2}\sqrt{(x+y)}=\dfrac{x-1}{\sqrt{(x-y)}},\\ \text{and } x^2+y^2 : xy :: 34 : 15,\end{array}\right\}$ to find the values of x and y.

$$\text{Ans. }\begin{cases}x=5,\\ y=3.\end{cases}$$

37. Given $\left.\begin{array}{r}x^4y^3-x^3y^4=216,\\ \text{and } x^2y-xy^2=6,\end{array}\right\}$ to find the values of x and y.

$$\text{Ans. }\begin{cases}x=3, \text{ or } -2,\\ y=2, \text{ or } -3.\end{cases}$$

38. Given $\left.\begin{array}{r}x^2+x\sqrt[3]{(xy^2)}=208,\\ \text{and } y^2+y\sqrt[3]{(x^2y)}=1053,\end{array}\right\}$ to find the values of x and y.

$$\text{Ans. }\begin{cases}x=\pm 8,\\ y=\pm 27.\end{cases}$$

39. Given $\left.\begin{array}{r}x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=1009,\\ \text{and } x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3=582193,\end{array}\right\}$ to find the values of x and y.

$$\text{Ans. }\begin{cases}x=81, \text{ or } 16,\\ y=16, \text{ or } 81.\end{cases}$$

40. Given $x^2 + y^2 + xy \,.\, (x + y) = 68,$ and $x^3 + y^3 - 3x^2 = 12 + 3y^2,$ $\Big\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 4, \text{ or } 2, \\ y = 2, \text{ or } 4. \end{cases}$$

41. Given $xy \,.\, (x + y) = 84,$ and $x^2y^2 \,.\, (x^2 + y^2) = 3600,$ $\Big\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 4, \text{ or } 3, \\ y = 3, \text{ or } 4. \end{cases}$$

42. Given $\dfrac{x^2 + xy + y^2}{x + y} = 7,$ and $\dfrac{x^2 - xy + y^2}{x - y} = 9,$ $\Big\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 6, \\ y = 3. \end{cases}$$

43. Given $\sqrt{\left(\dfrac{x}{4} + 3\right)} - \sqrt{\left(\dfrac{x}{4} - 3\right)} = \sqrt{\left(\dfrac{2x}{3}\right)}$, to find the values of x.

$$\text{Ans. } x = \pm\, 9\sqrt{2}.$$

44. Given $\sqrt{x} - \sqrt{y} = \sqrt{x} \,.\, (\sqrt{x} + \sqrt{y}),$ and $(x + y)^2 = 2 \,.\, (x - y)^2,$ $\Big\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 1, \\ y = (3 - 2\sqrt{2})^2. \end{cases}$$

45. Given $\dfrac{x^{(m-n)^2} + x^{-4mn}}{x^{(m-n)^2} - x^{-4mn}} = a^{\frac{r}{s}}$, to find the value of x.

$$\text{Ans. } x = \left(\frac{a^{\frac{r}{s}} + 1}{a^{\frac{r}{s}} - 1}\right)^{\frac{1}{(m+n)^2}}.$$

46. Given $\left.\begin{aligned} & y + 3\sqrt[3]{y}\,.\left\{\sqrt[3]{(a+bx)} - \sqrt[3]{y}\right\}.\sqrt[3]{(a+bx)} = 2a, \\ & \text{and } \frac{y - 3\sqrt[3]{y}\,.\sqrt[3]{(a^2 - b^2x^2)}}{\sqrt[3]{y} - \sqrt[3]{(a+bx)}} = 2a\,.\sqrt[3]{(a-bx)}, \end{aligned}\right\}$

to find the values of x and y.

Ans. $\begin{cases} x = \dfrac{a-1}{b}, \\ y = \{\sqrt[3]{(2a+1)} + 1\}^2. \end{cases}$

IV. *Adfected Quadratics involving only one unknown Quantity.*

1. Given $x^2 + 4x = 140$, to find the values of x.

   Ans. $x = 10$, or -14.

2. Given $x^2 - 6x + 8 = 80$, to find the values of x.

   Ans. $x = 12$, or -6.

3. Given $x^2 - 10x + 17 = 1$, to find the values of x.

   Ans. $x = 8$, or 2.

4. Given $x^2 - x - 40 = 170$, to find the values of x.

   Ans. $x = 15$, or -14.

5. Given $3x^2 - 9x - 4 = 80$, to find the values of x.

   Ans. $x = 7$, or -4.

6. Given $7x^2 - 21x + 13 = 293$, to find the values of x.

   Ans. $x = 8$, or -5.

7. Given $\frac{x^2}{3} + \frac{4x}{5} - 19 = 15\frac{1}{5}$, to find the values of x.

   Ans. $x = 9$, or $-\frac{57}{5}$.

8. Given $\frac{2x^2}{3} + 3\frac{1}{4} = \frac{x}{2} + 8$, to find the values of x.

Ans. $x = 3$, or $-\frac{9}{4}$.

9. Given $x + 4 + \frac{7x - 8}{x} = 13$, to find the values of x.

Ans. $x = 4$, or -2.

10. Given $4x - \frac{36 - x}{x} = 46$, to find the values of x.

Ans. $x = 12$, or $-\frac{3}{4}$.

11. Given $16 - \frac{5 - x}{2} = \frac{9 + 3x}{x} + 3x$, to find the values of x.

Ans. $x = 3$, or $\frac{6}{5}$.

12. Given $\frac{x + 3}{2} + \frac{16 - 2x}{2x - 5} = 5\frac{1}{2}$, to find the values of x.

Ans. $x = 5$, or $\frac{69}{10}$.

13. Given $14 + 4x - \frac{x + 7}{x - 7} = 3x + \frac{9 + 4x}{3}$, to find the values of x.

Ans. $x = 9$, or 28.

14. Given $\frac{x + 4}{3} - \frac{7 - x}{x - 3} = \frac{4x + 7}{9} - 1$, to find the values of x.

Ans. $x = 21$, or 5.

15. Given $\frac{15 - x}{4} - \frac{12 - 3x}{4x - 5} = 7x - \frac{23x + 60}{7}$, to find the values of x.

Ans. $x = 3$, or $\frac{229}{148}$.

16. Given $\frac{x + 11}{x} + \frac{9 + 4x}{x^2} = 7$, to find the values of x.

Ans. $x = 3$, or $-\frac{1}{2}$.

17. Given $\frac{2x + 9}{9} + \frac{4x - 3}{4x + 3} = 3 + \frac{3x - 16}{18}$, to find the values of x.

Ans. $x = 6$, or $-\frac{19}{4}$.

18. Given $\frac{x}{x + 60} = \frac{7}{3x - 5}$, to find the values of x.

Ans. $x = 14$, or -10.

19. Given $\frac{3x - 7}{x} + \frac{4x - 10}{x + 5} = 3\frac{1}{2}$, to find the values of x.

Ans. $x = 7$, or $-\frac{10}{7}$.

20. Given $\frac{x + 2}{x - 1} - \frac{4 - x}{2x} = 2\frac{1}{3}$, to find the values of x.

Ans. $x = 3$, or $-\frac{4}{5}$.

21. Given $\frac{8x}{x + 2} - 6 = \frac{20}{3x}$, to find the values of x.

Ans. $x = 10$, or $-\frac{2}{3}$.

22. Given $\frac{40}{x - 5} + \frac{27}{x} = 13$, to find the values of x.

Ans. $x = 9$, or $\frac{15}{13}$.

23. Given $\frac{5x-12}{9} + \frac{3x-24}{4x-12} = 9 - \frac{7x-34}{15}$, to find the values of x.

ANS. $x = 12$, or $\frac{477}{184}$.

24. Given $\frac{2x}{x-4} + \frac{2x-5}{x-3} = 8\frac{1}{3}$, to find the values of x.

ANS. $x = 6$, or $\frac{40}{13}$.

25. Given $\frac{2x+3}{10-x} = \frac{2x}{25-3x} - 6\frac{1}{2}$, to find the values of x.

ANS. $x = 8$, or $13\frac{22}{31}$.

26. Given $\frac{4x-5}{x} - \frac{3x-7}{3x+7} = \frac{9x+23}{13x}$, to find the values of x.

ANS. $x = 2$, or $-\frac{154}{45}$.

27. Given $2x + 18 - \frac{8x^2+16}{4x+7} = 27 - \frac{12x-11}{2x-3}$, to find the values of x.

ANS. $x = 8$, or 5.

28. Given $\frac{x}{x+9} + \frac{5}{2x+18} = \frac{x - \frac{x^2+20}{x+8}}{2}$, to find the values of x.

ANS. $x = 4$, or $-\frac{55}{6}$.

29. Given $\frac{x+4}{x+6} + \frac{5}{2x+4} = \frac{3x+7}{3x+4}$, to find the values of x.

ANS. $x = 8$, or $-\frac{2}{3}$.

30. Given $\frac{4}{2x+3} + \frac{3x+6}{5x+18} = \frac{3x+5}{5x}$, to find the values of x.

Ans. $x = 6$, or $-\frac{45}{2}$.

31. Given $\frac{8}{9+5x} + \frac{8x-17}{2+4x} = \frac{4x+3}{2x+12}$, to find the values of x.

Ans. $x = 3$, or $-\frac{283}{137}$.

32. Given $\frac{12}{5-x} + \frac{8}{4-x} = \frac{32}{x+2}$, to find the values of x.

Ans. $x = 2$, or $\frac{58}{13}$.

33. Given $\frac{2x-1}{3-x} = \frac{8-x^2}{2x-2} + \frac{x}{2}$, to find the values of x.

Ans. $x = 2$, or $-\frac{11}{3}$.

34. Given $\frac{3}{6x-x^2} + \frac{6}{x^2+2x} = \frac{11}{5x}$, to find the values of x.

Ans. $x = 3$, or $\frac{26}{11}$.

35. Given $\frac{4x^2+7x}{19} + \frac{5x-x^2}{3+x} = \frac{4x^2}{9}$, to find the values of x.

Ans. $x = 3$, or $-\frac{87}{10}$.

36. Given $\frac{x^4+2x^3+8}{x^2+x-6} = x^2+x+8$, to find the values of x.

Ans. $x = 4$, or $-\frac{14}{3}$.

37. Given $\frac{x+12}{x} + \frac{x}{x+12} = 5\frac{3}{15}$, to find the values of x.

Ans. $x = 3$, or -15.

38. Given $\sqrt{(4x+5)} \times \sqrt{(7x+1)} = 30$, to find the values of x.

$$\text{Ans. } x = 5, \text{ or } -\frac{179}{28}.$$

39. Given $\dfrac{\sqrt{x}+9}{\sqrt{x}} = \dfrac{\sqrt{9x} - 3\frac{4}{5}}{9-\sqrt{x}}$, to find the values of x.

$$\text{Ans. } x = 25, \text{ or } \frac{6561}{400}.$$

40. Given $\dfrac{x+\sqrt{x}}{x-\sqrt{x}} = \dfrac{x^2-x}{4}$, to find the values of x.

$$\text{Ans. } x = 4, \text{ or } 1, \text{ or } \frac{-3 \pm \sqrt{-7}}{2}.$$

41. Given $\dfrac{x-\sqrt{(x+1)}}{x+\sqrt{(x+1)}} = \dfrac{5}{11}$, to find the values of x.

$$\text{Ans. } x = 8, \text{ or } -\frac{8}{9}.$$

42. Given $5 \cdot \dfrac{3x-1}{1+5\sqrt{x}} + \dfrac{2}{\sqrt{x}} = 3\sqrt{x}$, to find the values of x.

$$\text{Ans. } x = 4, \text{ or } \frac{1}{9}.$$

43. Given $\sqrt{x^5} - \dfrac{40}{\sqrt{x}} = 3x$, to find the values of x.

$$\text{Ans. } x = 4, \text{ or } (-5)^{\frac{2}{3}}.$$

44. Given $x^{\frac{4}{3}} + 7x^{\frac{2}{3}} = 44$, to find the values of x.

$$\text{Ans. } x = \pm 8, \text{ or } \pm(-11)^{\frac{3}{2}}.$$

45. Given $4x^{\frac{1}{3}} + x^{\frac{1}{6}} = 39$, to find the values of x.

$$\text{Ans. } x = 729, \text{ or } \left(\frac{13}{4}\right)^6.$$

46. Given $3x^6 + 42x^3 = 3321$, to find the values of x.

Ans. $x = 3$, or $-\sqrt[3]{41}$.

47. Given $\frac{8}{x^3} + 2 = \frac{17}{x^{\frac{3}{2}}}$, to find the values of x.

Ans. $x = 4$, or $\sqrt[3]{\frac{1}{4}}$.

48. Given $x^{\frac{7}{3}} + \frac{41\sqrt[3]{x}}{x} = \frac{97}{\sqrt[3]{x^2}} + x^{\frac{5}{6}}$, to find the values of x.

Ans. $x = 4$, or $(-7)^{\frac{2}{3}}$.

49. Given $\sqrt[6]{\frac{1}{x^4}} + \sqrt[3]{\frac{1}{x}} = \frac{3 - \sqrt[3]{x^2}}{x}$, to find the values of x.

Ans. $x = 1$, or $-\frac{27}{8}$.

50. Given $3x^n\sqrt[3]{x^n} - \frac{4x^n}{\sqrt[3]{x^n}} = 4$, to find the values of x.

Ans. $x = (8)^{\frac{1}{2n}}$, or $\left(-\frac{8}{27}\right)^{\frac{1}{2n}}$.

51. Given $\frac{3\sqrt{x} - x^{\frac{3}{2}}}{x + 2} = \frac{1\frac{2}{3} + 3\sqrt{x} - 2x}{2\sqrt{x} - 3}$, to find the values of x.

Ans. $x = 4$, or $\frac{1}{25}$.

52. Given $2x^{\frac{3}{2}} \,.\, (x^3 + a^3)^{\frac{1}{2}} = 2x^2 \,.\, (x + 2a) + a^2 \,.\, (x - a)$, to find the values of x.

Ans. $x = \frac{a}{2}$, or $-a$.

53. Given $adx - acx^2 = bcx - bd$, to find the values of x.

Ans. $x = \frac{d}{c}$, or $-\frac{b}{a}$.

54. Given $\frac{a^2x^2}{b^2} - \frac{2ax}{c} + \frac{d^2}{c^2} = 0$, to find the values of x.

$$\text{Ans. } x = \frac{b}{a} \times \frac{b \pm \sqrt{(b^2 - d^2)}}{c}.$$

55. Given $9a^4b^4x^2 - 6a^3b^2x = b^2$, to find the values of x.

$$\text{Ans. } x = \frac{a^2 \pm \sqrt{(a^2 + b^2)}}{3a^2b^2}.$$

56. Given $(a + b) \cdot x^2 = cx + \frac{ac}{a+b}$, to find the values of x.

$$\text{Ans. } x = \frac{c \pm \sqrt{(c^2 + 4ac)}}{2 \cdot (a + b)}.$$

57. Given $3\sqrt{(112 - 8x)} = 19 + \sqrt{(3x + 7)}$, to find the values of x.

$$\text{Ans. } x = 6, \text{ or } \frac{7398}{625}.$$

58. Given $\sqrt{(2x + 7)} + \sqrt{(3x - 18)} = \sqrt{(7x + 1)}$, to find the values of x.

$$\text{Ans. } x = 9, \text{ or } -\frac{18}{5}.$$

59. Given $7 \cdot \sqrt{\left(\frac{3x}{2} - 5\right)} - \sqrt{\left(\frac{x}{5} + 46\right)} = \frac{7}{4}\sqrt{(10x + 56)}$, to find the values of x.

$$\text{Ans. } x = 20, \text{ or } \frac{14568980}{2874649}.$$

60. Given

$$\frac{16 - 4\sqrt{x}}{8 - 3\sqrt{x}} = \frac{88 + 33\sqrt{x}}{4 + \sqrt{x}} + \frac{x^2 - 5x + 11}{(8 - 3\sqrt{x}) \cdot (4 + \sqrt{x})},$$

to find the values of x.

$$\text{Ans. } x = 93, \text{ or } 7.$$

61. Given

$$\frac{54 - 9\sqrt{x}}{x + 2\sqrt{x}} = \frac{23x - 46\sqrt{x}}{6 + \sqrt{x}} + \frac{7x^2 - 3x + 4}{(x + 2\sqrt{x}) \times (6 + \sqrt{x})},$$

to find the values of x.

ANS. $x = 5$, or $-\frac{32}{15}$.

62. Given $x + \sqrt{x} : x - \sqrt{x} :: 3\sqrt{x} + 6 : 2\sqrt{x}$, to find the values of x.

ANS. $x = 9$, or 4.

63. Given $x^2 + 11 + \sqrt{(x^2 + 11)} = 42$, to find the values of x.

ANS. $x = \pm 5$, or $\pm\sqrt{38}$.

64. Given $(x - 5)^3 - 3 \, . \, (x - 5)^{\frac{3}{2}} = 40$, to find the values of x.

ANS. $x = 9$, or $(-5)^{\frac{2}{3}} + 5$.

65. Given $x + \sqrt{(x + 6)} = 2 + 3\sqrt{(x + 6)}$, to find the values of x.

ANS. $x = 10$, or -2.

66. Given $(x^2 + 5)^2 - 4x^2 = 160$, to find the values of x.

ANS. $x = \pm 3$, or $\pm\sqrt{(-15)}$.

67. Given $x^2 - 7x + \sqrt{(x^2 - 7x + 18)} = 24$, to find the values of x.

ANS. $x = 9$, or -2, or $\frac{7 \pm \sqrt{(173)}}{2}$.

68. Given $9x - 4x^2 + \sqrt{(4x^2 - 9x + 11)} = 5$, to find the values of x.

ANS. $x = 2$, or $\frac{1}{4}$, or $\frac{9 \pm \sqrt{(-31)}}{8}$.

69. Given $x^2 + \sqrt{(5x + x^2)} = 42 - 5x$, to find the values of x.

ANS. $x = 4$, or -9, or $\frac{-5 \pm \sqrt{221}}{2}$.

70. Given $\frac{2}{(x+2)^{\frac{3}{2}}} + \frac{\sqrt{(x+2)}}{2} = \frac{17}{4\sqrt{(x+2)}}$, to find the values of x.

Ans. $x = 6$, or $-\frac{3}{2}$.

71. Given $\frac{x}{x+4} + \frac{4}{\sqrt{(x+4)}} = \frac{21}{x}$, to find the values of x.

Ans. $x = 12$, or -3, or $\frac{49 \pm \sqrt{(3185)}}{2}$.

72. Given $\frac{7+x}{7-x} + \frac{7-x}{7+x} = \frac{29}{10}$, to find the values of x.

Ans. $x = 7$, or -3.

73. Given $\frac{3x+5}{3x-5} - \frac{3x-5}{3x+5} = \frac{135}{176}$, to find the values of x.

Ans. $x = 9$, or $-\frac{25}{81}$.

74. Given $x + \sqrt{x} + 2 = \frac{x^3 + x - 4}{\sqrt{x}}$, to find the values of x.

Ans. $x = 4$, or 1.

75. Given $\frac{x^2}{(x^2-4)^2} + \frac{6}{x^2-4} = \frac{351}{25x^2}$, to find the values of x.

Ans. $x = \pm 3$, or $\pm\sqrt{\frac{39}{11}}$.

76. Given $\left(x + \frac{8}{x}\right)^2 + x = 42 - \frac{8}{x}$, to find the values of x.

Ans. $x = 4$, or 2, or $\frac{-7 \pm \sqrt{17}}{2}$.

77. Given $x + 4 - 2\sqrt{\left(\frac{x+4}{x-4}\right)} = \frac{3}{x-4}$, to find the values of x.

ANS. $x = \pm 5$, or $\pm\sqrt{17}$.

78. Given $\sqrt{\left(12 - \frac{12}{x^2}\right)} + \sqrt{\left(x^2 - \frac{12}{x^2}\right)} = x^2$, to find the values of x.

ANS. $x = \pm 2$, or $\pm\sqrt{-3}$.

79. Given $\sqrt{\{(x-1).(x-2)\}} + \sqrt{\{(x-3).(x-4)\}} = \sqrt{2}$, to find the values of x.

ANS. $x = 3$, or 2.

80. Given $x^4 . \left(1 + \frac{1}{3x}\right)^2 - (3x^2 + x) = 70$, to find the values of x.

ANS. $x = 3$, or $-\frac{10}{3}$, or $\frac{-1 \pm \sqrt{(-251)}}{6}$.

81. Given $x^2 - \frac{5x}{2} + 15 = \frac{25x^2}{16} - \frac{64}{x^2}$, to find the values of x.

ANS. $x = 4$, or -8, or $\frac{-2 \pm 2\sqrt{(-71)}}{9}$.

82. Given $\frac{35\frac{5}{7}}{\sqrt{(x^4 - 9x^2)}} + \frac{\sqrt{(x^2 - 9)}}{7x} = \frac{19}{2x}$, to find the values of x.

ANS. $x = \pm 5$, or $\pm\frac{\sqrt{(15661)}}{2}$.

83. Given $\frac{1}{x^2 + 11x - 8} + \frac{1}{x^2 + 2x - 8} + \frac{1}{x^2 - 13x - 8} = 0$, to find the values of x.

ANS. $x = \pm 8$, or ± 1.

84. Given $3 \,.\, \{(x-1)^2 - x\}^2 + 2x = 341 + 2 \,.\, (x-1)^2$, to find the values of x.

Ans. $x = 5$, or -2; or $\dfrac{3\sqrt{3} \pm \sqrt{(-109)}}{2\sqrt{3}}$.

85. Given $x^2 - 2x^{\frac{3}{2}} + 2x - \sqrt{x} = 6$, to find the values of x.

Ans. $x = 4$, or 1, or $\dfrac{-5 \pm \sqrt{(-11)}}{2}$.

86. Given $x^4 + \dfrac{13x^3}{3} - 39x = 81$, to find the values of x.

Ans. $x = \pm 3$, or $\dfrac{-13 \pm \sqrt{(-155)}}{6}$.

87. Given $x^2 - 2x + 4 = 2\sqrt{(x^3 - 1)}$, to find the values of x.

Ans. $x = 4 \pm \sqrt{6}$, or $\pm \sqrt{-2}$.

88. Given $\dfrac{x^2 + 8}{6} - \dfrac{4x + 6}{x^2} = \dfrac{8 - \dfrac{2}{x^2}}{3}$, to find the values of x.

Ans. $x = 4$, or -2; or $-1 \pm \sqrt{(-3)}$.

89. Given $\sqrt{x} - \dfrac{8}{x} = \dfrac{7}{\sqrt{x} - 2}$, to find the values of x.

Ans. $x = 16$, or 1, or $\dfrac{1 \pm 3\sqrt{(-7)}}{2}$.

90. Given $x - \dfrac{8}{\sqrt{x}} - \dfrac{2}{x} = 5 \,.\, \left\{1 + \dfrac{2}{x}\right\}$, to find the values of x.

Ans. $x = 9$, or 4, or $\dfrac{-3 \mp \sqrt{-7}}{2}$.

91. Given $4x^4 + \dfrac{x}{2} = 4x^3 + 33$, to find the values of x.

Ans. $x = 2$, or $-\dfrac{3}{2}$; or $\dfrac{1 \pm \sqrt{(-43)}}{4}$.

92. Given $x \,.\, (\sqrt{x} + 1)^2 = 102 \,.\, (x + \sqrt{x}) - 2576$, to find the values of x.

ANS. $x = 49$ or 64, or $\dfrac{93 \mp \sqrt{185}}{2}$.

93. Given $(x - 2)^2 - 6x^{\frac{1}{2}} \,.\, (x - 2) = 24 - 14x + 15x^{\frac{1}{2}}$, to find the values of x.

ANS. $x = 16$, or 1; or $\dfrac{\pm 3\sqrt{(-11)} - 1}{2}$.

94. Given $(4x + 1)^2 + 4x^{\frac{1}{2}} \,.\, (4x + 1) = 1912 - (10x + 3x^{\frac{1}{2}})$, to find the values of x.

ANS. $x = 9$, or $\dfrac{49}{4}$; or $\dfrac{-90 \mp \sqrt{(-181)}}{8}$.

95. Given $x^2 - \dfrac{27x}{4} + 25 = 7\sqrt{x} \,.\, (5 - x)$, to find the values of x.

ANS. $x = 4$, or $\dfrac{25}{4}$, or $\dfrac{209 \mp 13\sqrt{249}}{8}$.

96. Given $8x^2 - 13 = \dfrac{3x}{2} + \sqrt{(6x^3 + 52x^2)}$, to find the values of x.

ANS. $x = 2$, or $-\dfrac{13}{8}$; or $\dfrac{3 \pm \sqrt{(3337)}}{64}$.

97. Given $4x^2 + 21x + 8x^{\frac{1}{2}}\sqrt{(7x^2 - 5x)} = 207 - \dfrac{4x^2}{3}$, to find the values of x.

ANS. $x = 3$, or $\dfrac{207}{16}$; or $\dfrac{-129 \pm 3\sqrt{(-2567)}}{32}$.

98. Given $2x\sqrt{(1 - x^4)} = a \,.\, (1 + x^4)$, to find the values of x.

ANS. $x = \pm\sqrt{(-p \pm \sqrt{p^2 - 1})}$, where $p = \dfrac{1 \mp \sqrt{(1 - a^4)}}{a^2}$.

99. Given $\dfrac{1+x^4}{(1+x)^4} = \frac{1}{2}$, to find the values of x.

Ans. $x = 1 \pm \sqrt{3} \pm \sqrt{3} \,.\, \sqrt{(\sqrt{3} \pm 2)}$.

100. Given $\left(x - \dfrac{1}{3}\right)^2 - \dfrac{25}{9} = \dfrac{3x^2 + \dfrac{4}{9}}{2 \,.\left(x - \dfrac{1}{3}\right) + \sqrt{\left\{x \,.\left(x - \dfrac{8}{3}\right)\right\}}}$, to find the values of x.

Ans. $x = 3$, or $-\dfrac{1}{3}$; or $\dfrac{4 \pm 2\sqrt{13}}{3}$.

101. Given $\dfrac{2x + \sqrt{x}}{2x - \sqrt{x}} = 3\frac{7}{15} - 3 \,.\, \dfrac{2x - \sqrt{x}}{2x + \sqrt{x}}$, to find the values of x.

Ans. $x = 4$, or $\dfrac{49}{16}$.

102. Given $a^2 b^2 x^{\frac{1}{n}} - 4 \,.\, (ab)^{\frac{3}{2}} \,.\, x^{\frac{m+n}{2mn}} = (a - b)^2 \,.\, x^{\frac{1}{m}}$, to find the values of x.

Ans. $x = \left\{\dfrac{(\sqrt{a} + \sqrt{b})^2}{ab}\right\}^{\frac{2mn}{m-n}}$, or $\left\{\dfrac{-(\sqrt{a} - \sqrt{b})^2}{ab}\right\}^{\frac{2mn}{m-n}}$.

103. Given $\sqrt[2pq]{x^{p+q}} - \frac{1}{2} \,.\, \dfrac{a^2 - b^2}{a^2 + b^2} \,.\, \{\sqrt[p]{x} + \sqrt[q]{x}\} = 0$, to find the values of x.

Ans. $x = \left(\dfrac{a \pm b}{a \mp b}\right)^{\frac{2pq}{p-q}}$.

104. Given $\dfrac{n-1}{n+1} \,.\, \dfrac{a^4 + a^2x^2 + x^4}{a^4 - a^2x^2 + x^4} = 2 - \dfrac{1}{n} \,.\left(\dfrac{ax}{a^2 - x^2}\right)^2$, to find the values of x.

Ans. $x = a\,\{\pm \sqrt{(r^2 + 1)} - r\}$;

where $r = \pm \frac{1}{2} \sqrt{\left\{-p \pm \sqrt{\left(p^2 + \dfrac{(n-1) \,.\, (2n-1)}{n \,.\, (n+1)}\right)}\right\}}$,

and $p = \dfrac{n^2 + 6n + 1}{2n \,.\, (n+1)}$.

105. Given $x - 2\sqrt{(x + 2)} = 1 + \sqrt[4]{(x^2 - 3x + 2)}$, to find the values of x.

$$\text{Ans. } x = 9 \pm 4\sqrt{7}.$$

106. Given $(1-x) \cdot \sqrt{\left\{a \cdot \left(1 + \frac{1}{x}\right) - 2\right\}} = \sqrt{(x+1)} + \sqrt{(3x-1)}$, to find the values of x.

$$\text{Ans. } x = \frac{\pm\sqrt{(a+1)} - 1}{\pm\sqrt{(a-1)} + 1}.$$

V. *Adfected Quadratics involving two unknown Quantities.*

1. Given $\left.\begin{aligned} x + 4y &= 14, \\ \text{and } y^2 + 4x &= 2y + 11, \end{aligned}\right\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = -46, \text{ or } 2, \\ y = 15, \text{ or } 3. \end{cases}$$

2. Given $\left.\begin{aligned} 2x + 3y &= 118, \\ \text{and } 5x^2 - 7y^2 &= 4333, \end{aligned}\right\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 35, \text{ or } -\dfrac{3899}{17}, \\ y = 16, \text{ or } \dfrac{3268}{17}. \end{cases}$$

3. Given $\left.\begin{aligned} \frac{2x + 7y}{4x} &= 2y - \frac{51 + 2x}{10}, \\ \text{and } \frac{4x + 3y}{16} &= y - 2, \end{aligned}\right\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 5, \text{ or } -\dfrac{56}{27}, \\ y = 4, \text{ or } \dfrac{640}{351}. \end{cases}$$

4. Given $\left.\begin{array}{l}\dfrac{4xy+3y-3}{5x}-1=\dfrac{4y+3x-2}{5}-\dfrac{18-x}{3}, \\ \text{and } \dfrac{3x+y}{7}=\dfrac{3x-5y}{3}+2,\end{array}\right\}$

to find the values of x and y.

Ans. $\begin{cases} x = 6, \text{ or } -\dfrac{3}{266}, \\ y = 3, \text{ or } \dfrac{2784}{2527}. \end{cases}$

5. Given $\left.\begin{array}{l} x^2 - y^2 = a^2, \\ (x+y+b)^2 + (x-y+b)^2 = 2c^2, \end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = \dfrac{-b \pm \sqrt{(2a^2 - b^2 + 2c^2)}}{2}, \\ y = \pm \sqrt{\left(\dfrac{c^2 - a^2 \mp \sqrt{(2a^2 - b^2 + 2c^2)}}{2}\right)}. \end{cases}$

6. Given $\left.\begin{array}{l} x^2 + y : x^2 - y :: 9 : 7, \\ \text{and } 1 + x^2 : y + 4 :: 5y + 7 : 3y, \end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 4, \text{ or } \pm 4\sqrt{\left(-\dfrac{7}{19}\right)}, \\ y = 2, \text{ or } -\dfrac{14}{19}. \end{cases}$

7. Given $\left.\begin{array}{l} x^2 + 2x^3y = 441 - x^4y^2, \\ \text{and } xy = 3 + x, \end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 3, \text{ or } -7, \text{ or } -2 \pm \sqrt{(-17)}, \\ y = 2, \text{ or } \dfrac{4}{7}, \text{ or } \dfrac{5 \mp \sqrt{(-17)}}{7}. \end{cases}$

8. Given $\left.\begin{array}{l} x^2 + 4y^2 = 256 - 4xy, \\ \text{and } 3y^2 - x^2 = 39, \end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 6, \text{ or } \pm 102, \\ y = \pm 5, \text{ or } \pm 59. \end{cases}$

9. Given $(x+y)^2 - 3y = 28 + 3x,$ and $2xy + 3x = 35,$ } to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 5, \text{ or } \frac{7}{2}, \text{ or } \frac{-5 \pm \sqrt{(-255)}}{4}, \\ y = 2, \text{ or } \frac{7}{2}, \text{ or } \frac{-11 \mp \sqrt{(-255)}}{4}. \end{cases}$$

10. Given $(2x - 4y)^2 + x - 2y = 5,$ and $x^2 - y^2 = 8,$ } to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 3, \text{ or } \frac{17}{3}, \\ y = 1, \text{ or } \frac{7}{3}. \end{cases}$$

11. Given $\frac{3}{4}\sqrt{(x-y)} = 1 + \frac{1}{\sqrt{(x-y)}},$ and $\sqrt{(x+y)} + \sqrt{(x-y)} = 5,$ } to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \frac{13}{2}, \\ y = \frac{5}{2}. \end{cases}$$

12. Given $x^2 + 10x + y = 119 - 2\sqrt{y} \times (x+5),$ and $x + 2y = 13,$ } to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 5, \text{ or } \frac{17}{2}, \text{ or } \frac{-69 \pm \sqrt{(241)}}{4}, \\ y = 4, \text{ or } \frac{9}{4}, \text{ or } \frac{121 \mp \sqrt{(241)}}{8}. \end{cases}$$

13. Given $\frac{x^4}{y^2} + \frac{2x^2}{y} = 9\frac{39}{49},$ and $x^2 + y^2 = 65,$ } to find the values of x and y.

Answer,

$$x = \pm 4, \text{ or } \pm \frac{4\sqrt{(-65)}}{7}; \text{ or } \pm \frac{\sqrt{(-450 \mp 30\sqrt{3410})}}{\sqrt{7}},$$

$$y = 7, \text{ or } -\frac{65}{7}; \text{ or } \frac{15 \pm \sqrt{(3410)}}{7}.$$

14. Given $x + y + \sqrt{(x + y)} = 6,$ and $x^2 + y^2 = 10,$ $\Big\}$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 3, \text{ or } 1; \text{ or } \dfrac{9 \pm \sqrt{-61}}{2}, \\ y = 1, \text{ or } 3; \text{ or } \dfrac{9 \mp \sqrt{-61}}{2}. \end{cases}$$

15. Given $x^2 + 4\sqrt{(x^2 + 3y + 6)} = 55 - 3y,$ and $6x - 7y = 16,$ $\Big\}$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 5, \text{ or } \dfrac{-53}{7}; \text{ or } \dfrac{-9 \pm \sqrt{(3895)}}{7}, \\ y = 2, \text{ or } -\dfrac{46}{7}; \text{ or } \dfrac{-70 \pm 6\sqrt{(3895)}}{49}. \end{cases}$$

16. Given $x^2 + 3x + y = 73 - 2xy,$ and $y^2 + 3y + x = 44,$ $\Big\}$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 4, \text{ or } 16; \text{ or } -12 \pm \sqrt{58}, \\ y = 5, \text{ or } -7; \text{ or } -1 \mp \sqrt{58}. \end{cases}$$

17. Given $\dfrac{y}{(x + y)^{\frac{3}{2}}} + \dfrac{\sqrt{(x + y)}}{y} = \dfrac{17}{4\sqrt{(x + y)}},$ and $x = y^2 + 2,$ $\Bigg\}$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 6, \text{ or } 3, \text{ or } \dfrac{9 \mp 3\sqrt{(-119)}}{32}, \\ y = 2, \text{ or } 1, \text{ or } \dfrac{-3 \pm \sqrt{(-119)}}{8}. \end{cases}$$

18. Given $y - y^{\frac{1}{2}} = 16 - x,$ and $28 - y = x + 4x^{\frac{1}{2}},$ $\Big\}$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 4, \text{ or } \left|\overline{\dfrac{58}{17}}\right|^2. \\ y = 16, \text{ or } \dfrac{784}{289}. \end{cases}$$

19. Given $x^4 + y^4 = 97,$ and $x + y = 5,$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 3, \text{ or } 2; \text{ or } \dfrac{5 \pm \sqrt{(-151)}}{2}, \\ y = 2, \text{ or } 3; \text{ or } \dfrac{5 \mp \sqrt{(-151)}}{2}. \end{cases}$$

20. Given $\sqrt{\left(\dfrac{3x - 2y}{2x}\right)} + \sqrt{\left(\dfrac{2x}{3x - 2y}\right)} = 2,$ and $x^2 - 18 = x \cdot (4y - 9),$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 6, \text{ or } 3, \\ y = 3, \text{ or } \dfrac{3}{2}. \end{cases}$$

21. Given $x + 4\sqrt{x} + 4y = 21 + 8\sqrt{y} + 4\sqrt{xy},$ and $\sqrt{x} + \sqrt{y} = 6,$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 25, \text{ or } \dfrac{25}{9}, \\ y = \ \ 1, \text{ or } \dfrac{169}{9}. \end{cases}$$

22. Given $3x + \frac{4}{3}\sqrt{(xy^2 + 9x^2y)} = (x - \frac{1}{3}) \cdot y,$ and $6x + y : y :: x + 5 : 3,$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 4, \\ y = 12. \end{cases}$$

23. Given $x + y = 5,$ and $(x^2 + y^2) \times (x^3 + y^3) = 455,$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 3, \text{ or } 2; \text{ or } \dfrac{5}{2} \pm \dfrac{1}{2}\sqrt{\left(-\dfrac{103}{3}\right)}, \\ y = 2, \text{ or } 3; \text{ or } \dfrac{5}{2} \mp \dfrac{1}{2}\sqrt{\left(-\dfrac{103}{3}\right)}. \end{cases}$$

24. Given $x + y - \sqrt{\left(\frac{x+y}{x-y}\right)} = \frac{6}{x-y}$, and $x^2 + y^2 = 41$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = \pm 5, \text{ or } \pm 3\sqrt{\frac{5}{2}}, \\ y = \pm 4, \text{ or } \pm\sqrt{\frac{37}{2}}. \end{cases}$$

25. Given $\frac{x^4}{y^2} + \frac{y^4}{x^2} = 136\frac{1}{9} - 2xy$, and $x + 4 = 14 - y$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 6, \text{ or } 4;\ \text{or } 5 \pm 5\sqrt{\left(-\frac{13}{11}\right)}, \\ y = 4, \text{ or } 6;\ \text{or } 5 \mp 5\sqrt{\left(-\frac{13}{11}\right)}. \end{cases}$$

26. Given $\frac{x+y}{x-y} - \frac{x-y}{x+y} = 4\frac{4}{5}$, and $\sqrt{\left(\frac{x-y}{x^4}\right)} + \frac{1}{x} = \frac{4}{9\sqrt{(x-y)}}$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 3, \text{ or } \frac{45}{2};\ \text{or } \frac{3}{16}, \text{ or } \frac{45}{32}, \\ y = 2, \text{ or } -\frac{135}{4};\ \text{or } \frac{1}{8}, \text{ or } -\frac{135}{64}. \end{cases}$$

27. Given $\sqrt{(6\sqrt{x} + 6\sqrt{y})} + \frac{1}{2}\sqrt{x} = 9 - \frac{1}{2}\sqrt{y}$, and $x - y = 12$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 16, \text{ or } \frac{59536}{81}, \\ y = 4, \text{ or } \frac{58564}{81}. \end{cases}$$

28. Given $y^4 - 432 = 12xy^2$, and $y^2 = 12 + 2xy$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 2, \\ y = 6. \end{cases}$$

29. Given $\frac{4}{y^2} + \frac{4+y}{y} = \frac{8+4y}{x} + \frac{12y^2}{x^2}$, and $4y^2 - xy = x$, $\Big\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 2, \text{ or } -\frac{50}{3}, \\ y = 1, \text{ or } -\frac{5}{3}. \end{cases}$$

30. Given $\sqrt{\{(1+x)^2 + y^2\}} + \sqrt{\{(1-x)^2 + y^2\}} = 4$, and $(4 - x^2)^2 = 18 - 4y^2$, $\Big\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \pm 1, \text{ or } \pm \sqrt{10}, \\ y = \pm \frac{3}{2}, \text{ or } \pm 3\sqrt{-\frac{1}{2}}. \end{cases}$$

31. Given $\frac{x + \sqrt{x} + y}{x - \sqrt{x} + y} - \frac{\sqrt{x} - x - y}{\sqrt{x} + x + y} = 2\frac{9}{40}$, and $y^2 - \sqrt{xy^2} = \frac{4x}{9}$, $\Big\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 9, \text{ or } \frac{196}{9}, \text{ or } \frac{289}{9}, \text{ or } 16, \\ y = 4, \text{ or } \frac{-14}{9}, \text{ or } -\frac{68}{9}, \text{ or } \frac{4}{3}. \end{cases}$$

32. Given $\frac{x + \sqrt{(x^2 - y^2)}}{x - \sqrt{(x^2 - y^2)}} = 4\frac{1}{4} - \frac{x - \sqrt{(x^2 - y^2)}}{x + \sqrt{(x^2 - y^2)}}$, and $x \,.\, (x + y) = 52 - \sqrt{(x^2 + xy + 4)}$, $\Big\}$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \pm 5, \text{ or } \pm \frac{10}{\sqrt{3}}, \\ y = \pm 4, \text{ or } \pm \frac{8}{\sqrt{3}}. \end{cases}$$

33. Given $5y + \frac{\sqrt{(x^2 - 15y - 14)}}{5} = \frac{x^2}{3} - 36,$ and $\frac{x^2}{8y} + \frac{2x}{3} = \sqrt{\left(\frac{x^3}{3y} + \frac{x^2}{4}\right)} - \frac{y}{2},$ to find the values of x and y.

ANSWER,

$$x = 12, \text{ or } -\frac{19}{2}; \text{ or } \frac{25 \pm \sqrt{(41569)}}{20}; \text{ or } \frac{45 \pm \sqrt{(3849)}}{4},$$

$$y = 2, \text{ or } -\frac{19}{12}; \text{ or } \frac{25 \pm \sqrt{(41569)}}{120}; \text{ or } \frac{135 \pm 3\sqrt{(3849)}}{8}.$$

34. Given $\sqrt{\left(\frac{x + y^2}{4x}\right)} + \frac{y}{\sqrt{(y^2 + x)}} = \frac{y^2}{4} \cdot \sqrt{\left(\frac{4x}{y^2 + x}\right)},$ and $\frac{\sqrt{x} + \sqrt{(x - y - 1)}}{\sqrt{x} - \sqrt{(x - y - 1)}} = y + 1,$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 4, \text{ or } \frac{1}{4}; \text{ or } \frac{1 \mp 3\sqrt{(-7)}}{8}, \\ y = 2, \text{ or } -1; \text{ or } \frac{-7 \pm \sqrt{(-7)}}{2}. \end{cases}$$

35. Given $x^m a^n + y^n b^m = 2\,(ax)^{\frac{m}{2}} \cdot (by)^{\frac{n}{2}},$ and $xy = ab,$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = b^{\frac{2n}{m+n}} \cdot \left\{a^{\frac{m-n}{2}} \pm \sqrt{(a^{m-n} - b^{m-n})}\right\}^{\frac{2}{m+n}} \\ y = \dfrac{a\,b^{\frac{m-n}{m+n}}}{\left\{a^{\frac{m-n}{2}} \pm \sqrt{(a^{m-n} - b^{m-n})}\right\}^{\frac{2}{m+n}}}. \end{cases}$$

36. Given $\frac{x + y + \sqrt{(x^2 - y^2)}}{x + y - \sqrt{(x^2 - y^2)}} = \frac{9}{8y} \cdot (x + y),$ and $(x^2 + y)^2 + x - y = 2x \cdot (x^2 + y) + 506,$ to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 5, \text{ or } -\frac{23}{5}; \text{ or } \frac{1 \pm \sqrt{(-1209)}}{5}, \\ y = 3, \text{ or } -\frac{69}{25}; \text{ or } \frac{3 \pm 3\sqrt{(-1209)}}{25}. \end{cases}$$

37. Given $\frac{y}{x} \cdot \sqrt{\frac{x}{y}} + \frac{1}{2} \cdot \sqrt{\frac{x}{y}} \cdot \sqrt[4]{\frac{y^3}{x^3}} = 5,$ and $\frac{2x^2}{y} - \frac{x}{3\sqrt{y}} = \frac{1}{3},$ to find the values of x and y.

Ans. $\begin{cases} x = 4, \text{ or } \frac{16}{9}, \text{ or } \frac{625}{64}, \text{ or } \frac{625}{144}, \\ y = 64, \text{ or } \frac{256}{9}, \text{ or } \left(\frac{625}{32}\right)^2, \text{ or } \left(\frac{-625}{48}\right)^2. \end{cases}$

38. Given $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{61}{\sqrt{xy}} + 1,$ and $\sqrt[4]{x^3y} + \sqrt[4]{y^3x} = 78,$ to find the values of x and y.

Ans. $\begin{cases} x = 81, \text{ or } 16; \text{ or } -\frac{371}{12} \pm 3\sqrt{47}; \\ y = 16, \text{ or } 81; \text{ or } -\frac{371}{12} \mp 3\sqrt{47}. \end{cases}$

39. Given $x + \sqrt{(3y^2 - 11 + 2x)} = 7 + 2y - y^2,$ and $\sqrt{(3y - x + 7)} = \frac{x+y}{x-y},$ to find the values of x and y.

Ans. $\begin{cases} x = 4, \\ y = 2. \end{cases}$

40. Given $x^4 + y^4 = 1 + 2xy + 3x^2y^2,$ and $x^3 + y^3 = 2y^2x + 2y^2 + x + 1,$ to find the values of x and y.

Ans. $\begin{cases} x = 2, \text{ or } -1, \\ y = 1. \end{cases}$

41. Given $x^2y - 4 = 4x^{\frac{1}{2}}y - \frac{y^2}{4},$ $x^{\frac{3}{2}} - 3 = x^{\frac{1}{2}}y^{\frac{1}{2}} . (x^{\frac{1}{2}} - y^{\frac{1}{2}}),$ to find the values of x and y.

Ans. $\begin{cases} x = 1, \\ y = 4. \end{cases}$

42. Given $5 - 2\sqrt{(y+2)} = \frac{9x^2}{64} - \{\sqrt{x} - 3\sqrt{y}\}^2,$
and $\frac{7}{y} - 10\sqrt{\frac{x}{y}} = x - 16,$
to find the values of x and y.

ANS. $\begin{cases} x = 4, \\ y = \frac{1}{4}. \end{cases}$

43. Given $\frac{y}{2x} + \frac{2}{3} \cdot \frac{y - \sqrt{(x-1)}}{y^2 - 2\sqrt{(x^2-1)}} = \frac{\sqrt{(x+1)}}{x},$
and $\frac{1}{4} \cdot y^4 = y^2 x - 1,$
to find the values of x and y.

ANS. $\begin{cases} x = \frac{5}{4}, \text{ or } -\frac{85}{36}, \\ y = \pm 2, \text{ or } \pm\sqrt{\left(-\frac{85}{18} + \frac{\sqrt{5929}}{648}\right)}. \end{cases}$

44. Given $(x-2) \cdot y - \sqrt{xy} \cdot (y^2 - 1) = 2y^2 - x,$
and $\frac{1}{4}xy = \frac{\sqrt{xy} - 12}{xy - 18},$
to find the values of x and y.

ANS. $\begin{cases} x = 8, \text{ or } -12, \\ y = 2, \text{ or } -\frac{4}{3}. \end{cases}$

45. Given $3x - x\sqrt{\left(\frac{5x^2}{4} - 2y + 8\right)} = 2 - y,$
and $\frac{\sqrt{(x+y)}}{2x} - \frac{3}{4}x = \frac{2x - 3}{\sqrt{(x+y)}} - \frac{3y}{2x},$
to find the values of x and y.

ANS. $\begin{cases} x = 2, \text{ or } \frac{34}{31}, \\ y = 2, \text{ or } \frac{1446}{961}. \end{cases}$

46. Given $x^2 - y^2 = 3,$
and $(x^4 + y^4)^2 + x^2y^2 \cdot (x^2 - y^2)^2 + x^2 - y^2 = 328,$
to find the values of x and y.

$$\text{Ans.}\begin{cases} x = \pm 2, \text{ or } \pm \sqrt{-1}, \text{ or } \pm \sqrt{\left(\dfrac{3 \pm 2\sqrt{-13}}{2}\right)}, \\ y = \pm 1, \text{ or } \pm 2\sqrt{-1}, \text{ or } \pm \sqrt{\left(\dfrac{-3 \mp \sqrt{-13}}{2}\right)}. \end{cases}$$

47. Given $y^4 = x^2 \cdot (ay - bx),$
and $x^2 = ax - by,$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = a + br, \\ y = ar + br^2; \end{cases}$$

where $r = p \pm \sqrt{p^2 - 1},$

and $p = \dfrac{a - b \pm \sqrt{a^2 + 2ab + 5b^2}}{4b}.$

48. Given $\dfrac{3 + 2x^2 - 4x^4}{x^2 - 1} = y^2 \cdot (1 - 2y^2),$
and $(2x^2 - 1) \cdot (2y^2 - 1) = 3,$ to find the values of x and y.

$$\text{Ans.}\begin{cases} x = \pm \dfrac{\sqrt{5}}{2}, \text{ or } 0; \text{ or } \pm \tfrac{1}{2}\sqrt{\left(\dfrac{1 \pm \sqrt{33}}{2}\right)}, \\ y = \pm \sqrt{\dfrac{3}{2}}, \text{ or } \pm \sqrt{-1}; \text{ or } \pm \tfrac{1}{2}\sqrt{(5 \pm \sqrt{33})}. \end{cases}$$

49. Given $\dfrac{x^2y^2}{2} + 4 - 40y^2 = 140 - y^2 \cdot \sqrt{\left(x^2 - \dfrac{272}{y^2}\right)},$
and $x^2 - \dfrac{2}{y} \cdot \left(\dfrac{3}{y} + 15x\right) = \dfrac{30}{y^2} + \dfrac{5x}{y},$
to find the values of x and y.

$$\text{Ans.}\begin{cases} x = 9, \text{ or } -\dfrac{45}{4}, \\ y - 4, \text{ or } -\dfrac{16}{5}. \end{cases}$$

50. Given $\frac{y}{x} - \frac{9\sqrt{x}}{y} - \frac{81}{xy} = (2y + 9) \cdot \frac{\sqrt{x}}{y},$ and $\frac{\sqrt{y}}{x} + 3\sqrt{\frac{x}{y}} = \frac{9}{x\sqrt{y}} + \sqrt{x},$ to find the values of x and y.

ANS. $\begin{cases} x = 4, \\ y = 25. \end{cases}$

51. Given $\frac{\sqrt{(y^2 + 1)} + 1}{y} = \frac{\sqrt{(x + 9)} + 3}{\sqrt{x}},$ and $x \cdot (y + 1)^2 = 36 \cdot \left(y^2 + \frac{16}{9}\right),$ to find the values of x and y.

ANS. $\begin{cases} x = \frac{9}{4} \cdot \left\{1 \pm \sqrt{\frac{35}{3}}\right\}^2, \text{ or } \frac{9}{4} \cdot \left\{1 \pm \sqrt{\frac{-29}{3}}\right\}^2, \\ y = \frac{1}{2} \cdot \left(1 \pm \sqrt{\frac{35}{3}}\right), \text{ or } \frac{1}{2} \cdot \left(1 \pm \sqrt{\frac{-29}{3}}\right). \end{cases}$

52. Given $(x^6 + 1) \cdot y = (y^2 + 1) \cdot x^3,$ and $(y^6 + 1) \cdot x = 9 \cdot (x^2 + 1) \cdot y^3,$ to find the values of x and y.

ANS. $\begin{cases} x = a \pm \sqrt{(a^2 + 1)}, \text{ where } a = \pm \tfrac{1}{2}\sqrt{(\sqrt[3]{3} - 1)}, \\ y = b \pm \sqrt{(b^2 - 1)}, \text{ where } b = \tfrac{1}{2}\sqrt[3]{3} \cdot \sqrt{\{3 + \sqrt[3]{3}\}}. \end{cases}$

53. Given $\frac{2y^2 - 8\sqrt{x}}{\sqrt{x}} + \sqrt{(4y^2 - 16\sqrt{x})} = \frac{3\sqrt{x}}{2},$ and $\sqrt{x} + \sqrt{\{8 \cdot (y - \sqrt{x}) - 4\}} = y + 1,$ to find the values of x and y.

ANSWER,

$x = 4, \text{ or } \frac{4}{9}, \text{ or } \frac{4}{25}, \text{ or } -\frac{4}{3} \mp 16\sqrt{-\frac{13}{3}}, \text{ or } \frac{788 \pm 24\sqrt{644}}{25},$

$y = 3, \text{ or } \frac{5}{3}; \text{ or } \frac{7}{5}, \text{ or } -1; \text{ or } 1 \pm 2\sqrt{-\frac{13}{3}}, \text{ or } \frac{37 \pm \sqrt{644}}{5}.$

54. Given $\left.\begin{array}{l} \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}} \cdot (x^{\frac{2}{3}} - 1) + \frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \cdot (2x^{\frac{2}{3}} - 1) = \frac{4y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \cdot (y^{\frac{1}{3}} + x^{\frac{1}{3}}) \\ \qquad + \frac{3y}{x^{\frac{1}{3}}} + 2, \\ \text{and } \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}} - \frac{2x^{\frac{2}{3}}}{y} - \frac{2x^{\frac{1}{3}}}{y^{\frac{1}{3}}} = \frac{133}{36} \cdot \frac{1}{y^{\frac{2}{3}}} - \frac{2}{x^{\frac{1}{3}}} - \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}, \end{array}\right\}$

to find the values of x and y.

ANS. $\begin{cases} x = 27, \\ y = 8. \end{cases}$

VI. *Problems producing Simple Equations, involving only one unknown Quantity.*

1. WHAT number is that, from the treble of which if 18 be subtracted, the remainder is 6? *Ans.* 8.

2. What number is that, the double of which exceeds four-fifths of its half by 40? *Ans.* 25.

3. In fencing the side of a field, whose length was 450 yards, two workmen were employed; one of whom fenced 9 yards, and the other 6 *per* day. How many days did they work? *Ans.* 30.

4. A Mercer bought 4 pieces of silk, which together measured 50 yards; the second was twice, the third three times, and the fourth four times as long as the first. What were the respective lengths of the pieces?

Ans. 5, 10, 15, 20 yards.

5. A Farmer sold 13 bushels of barley at a certain price; and afterwards 17 bushels at the same rate; and at the second time received 36 shillings more than at the first. What was the price of a bushel?

Ans. 9 shillings.

6. A person bought 198 gallons of beer, which exactly filled 4 casks; the first held twice as much as the second, the second twice as much as the third, and the third three times as much as the fourth. How many gallons did each hold?

Ans. 108, 54, 27, and 9 gallons.

7. A Silversmith has 3 pieces of metal, which together weigh 48 ounces. The second weighs 12 ounces more than the first, and the third 9 ounces more than the second. What are their respective weights?

Ans. 5, 17, and 26 ounces.

8. A Vintner fills a cask, containing 96 gallons, with a mixture of brandy, wine, and water. There are 20 gallons of water more than of brandy, and 17 more of wine than of water. How many are there of each?

Ans. 13 gallons of brandy, 33 of water, and 50 of wine.

9. A Gentleman buys 4 horses; for the second of which he gives £12 more than for the first; for the third £6 more than for the second; and for the fourth £2 more than for the third. The sum paid for all was £230. How much did each cost?

Ans. 45, 57, 63, and 65 pounds.

10. A poor man had 6 children, the eldest of which could earn 7*d.* a week more than the second; the second 8*d.* more than the third; the third 6*d.* more than the fourth; the fourth 4*d.* more than the fifth; and the fifth 5*d.* more than the youngest. They altogether earned 10*s.* 10*d.* a week. How much could each earn a week?

Ans. 38, 31, 23, 17, 13, and 8 pence *per* week.

11. An express set out to travel 240 miles in 4 days, but in consequence of the badness of the roads, he found that he must go 5 miles the second day, 9 the third, and 14 the fourth day, less than the first. How many miles must he travel each day?

Ans. 67, 62, 58, and 53 miles.

12. There are 5 towns, in the order of the letters, A, B, C, D, E. From A to E is 80 miles. The distance between B and C is 10 miles more, between C and D is 15 miles less, and between D and E 17 miles more than the distance between A and B. What are the respective distances?

Ans. From A to B 17; from B to C 27; from C to D 2; and from D to E 34 miles.

13. A gentleman gave 27 shillings to two poor persons; but he gave 5 shillings more to one than to the other. What did he give to each?

Ans. 11, and 16 shillings.

14. What number is that, the treble of which is as much above 40, as its half is below 51?

Ans. 26.

15. Two workmen received the same sum for their labour; but if one had received 15 shillings more, and the other 9 shillings less, then one would have had just three times as much as the other. What did they receive?

Ans. 21 shillings each.

16. Two merchants entered into a speculation, by which one gained £54 more than the other. The whole gain was £49 less than three times the gain of the less. What were the gains?

Ans. £103, and £157.

17. The perimeter of a triangle is 75 feet, and the base is 11 feet longer than one of the sides, and 16 feet longer than the other. Required their respective lengths.

Ans. 34, 23, and 18 feet.

18. A company settling their reckoning at a tavern, pay 8 shillings each; but observe, that if there had been 4 more, they should only have paid 7 shillings each. How many were there?

Ans. 28.

19. Divide the number 46 into two such parts, that one of them being divided by 7 and the other by 3, the quotients may together be equal to 10.

Ans. 28 and 18.

20. A certain sum is to be raised upon two estates, one of which pays 19 shillings less than the other; and if 5 shillings be added to treble the less payment, it will be equal to twice the greater. What are the sums paid?

Ans. 33, and 52 shillings.

21. Having bought a certain quantity of brandy at 19 shillings a gallon, and a quantity of rum exceeding that of the brandy by 9 gallons at 15 shillings a gallon, I find that I paid one shilling more for the brandy than for the rum. How many gallons were there of each?

Ans. 34 of brandy, and 43 of rum.

22. Two persons, *A* and *B*, have each an annual income of £400. *A* spends every year £40 more than *B*, and at the end of 4 years, the amount of their savings is equal to one year's income of either. What does each spend annually?

Ans. £370, and £330, respectively.

23. A Draper sold two pieces of cloth, by one of which he lost £6 more than by the other; and his whole loss was £5 less than treble the less loss. What were the losses sustained by each piece?

Ans. £11, and £17.

24. A person engaged to reap a field of corn for 5 shillings an acre, but leaving 6 acres not reaped, he received £2. 10*s*. Of how many acres did the field consist?

Ans. 16.

25. In a naval engagement, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk.

Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist?

Ans. 32.

26. A Farmer hires a farm for £175. 15*s*. a year; part of which he is not allowed to plough: he gives £2 *per* acre for the arable, and for the rest, which was 5 acres less, he gives £1. 5*s*. *per* acre. How many acres were arable, and how many not?

Ans. 56 acres arable, 51 not.

27. A Cistern is filled in twenty minutes by three pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third, *per* minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

Ans. 22, 7, and 12 gallons.

28. A Fortress is garrisoned by 2600 men; and there are 9 times as many infantry, and three times as many artillery as cavalry. How many are there of each?

Ans. 200 cavalry, 600 artillery, and 1800 infantry.

29. *A* and *B* began to play; *A* with exactly four-ninths of the sum which *B* had. After winning £10, he found that they had each the same sum. What had each at first?

Ans. *A* had £16, and *B* £36.

30. A person has a certain number of horses at livery stables, and three times as many at grass. He keeps 15 in constant employment; and his whole number is seven times the number in the stables. Required the whole number.

Ans. 35.

31. Two men at the distance of 150 miles set out to meet each other, one goes 3 miles in the time the other goes 7. What part of the distance does each travel?

Ans. One 45, and the other 105 miles.

32. A Farm of 864 acres is divided between 3 persons. *C* has as many acres as *A* and *B* together; and the portions of *A*

and B are in the proportion of $5 : 11$. How many acres has each?

Ans. A has 135, B 297, C 432.

33. A charitable person distributed £5. 14*s*. amongst some poor women and children, giving to each woman 6 shillings, and to each child two; and the number of women was to the number of children as $4 : 7$. How many were relieved?

Ans. 12 women, and 21 children.

34. A and B begin trade, A with triple the stock of B. They each gain £50, which makes their stocks in the proportion of 7 to 3. What were their original stocks?

Ans. A's was £300, and B's £100.

35. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, which being increased respectively by 6 and 5, are in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$. Required the numbers.

Ans. 30 and 40.

36. A Farmer has a stack of hay, from which he sells a quantity which is to the quantity remaining in the proportion of 4 to 5. He then uses 15 loads, and finds that he has a quantity left which is to the quantity sold as 1 to 2. How many loads did the stack at first contain?

Ans. 45.

37. There are three pieces of cloth, whose lengths are in the proportion of 3, 5, and 7; and 6 yards being cut off from each, the whole quantity is diminished in the proportion of 20 to 17. Required the length of each piece at first.

Ans. 24, 40, and 56 yards.

38. The number of days that 4 workmen were employed were severally as the numbers 4, 5, 6, 7; their daily wages were the same, viz. 3 shillings, and the sum received by the first and

second was 36 shillings less than that received by the third and fourth. How much did each receive?

Ans. 36, 45, 54, and 63 shillings.

39. From two casks of equal size are drawn quantities, which are in the proportion of 6 to 7; and it appears that if 16 gallons less had been drawn from that which is now the emptier, only half as much would have been drawn from it as from the other. How many gallons were drawn from each?

Ans. 24 and 28.

40. On the enclosure of a parish, a proprietor had for his allotment two pieces of land, which were of the form of rectangular parallelograms. The longer sides of the parallelograms were in the ratio of 6 to 11, and the adjacent sides of the less as 3 to 2. The periphery of the less was 135 yards more than the longer side of the greater. Required the sides of the less, and the longer side of the greater.

Ans. The sides of the less were 90 and 60; and the longer side of the greater was 165 yards.

41. Two persons A and B travelling, each with £80, meet with robbers, who take from A twice as much as from B and £5 over, and leave A within £13 half as much as B. How much is taken from each?

Ans. 69, and 32 pounds.

42. A person distributes forty shillings amongst 50 people; giving to some nine-pence each, and to the rest fifteen pence. How many were there of each?

Ans. 45, and 5.

43. A person put out a certain sum to interest for $6\frac{1}{4}$ years, at 5 *per* cent. simple interest, and found that if he had put out the same sum for 12 years and 9 months at 4 *per* cent. he would have received £185 more. What was the sum put out?

Ans. £1000.

44. A regiment of militia containing 594 men, is to be raised from three towns, *A, B, C.* The contingents of *A* and *B* are in the proportion of three to five; and of *B* and *C* in the proportion of eight to seven. Required the numbers raised by each.

Ans. 144 by *A*, 240 by *B*, 210 by *C.*

45. Two persons, *A* and *B*, were partners. *A*'s money remained in the firm 6 years, and his gain was one-fourth of his principal; and *B*'s money, which was £50 less than *A*'s, had been in the firm 9 years, when they dissolved partnership; and it appeared that if *B* had gained £6. 5*s*. less, his gain and principal would have been to *A*'s gain and principal as 4 to 5. What was the principal of each?

Ans. £200, and £150.

46. The estate of a Bankrupt, valued at £21000, is to be divided amongst four creditors proportionably to what is due to them. The debts due to *A* and *B* are as 2 : 3; *B*'s claims and *C*'s are in the proportion of 4 : 5; and *C*'s and *D*'s in the proportion of 6 : 7. What sum must each receive?

Ans. A £3200, *B* £4800, *C* £6000, *D* £7000.

47. A Merchant bought wheat at the rate of £3. 10*s*. for 5 bushels. He afterwards bought some inferior, which was in quantity to the former as 3 to 4, at the rate of £3. 12*s*. for 8 bushels; and sold the whole for 10 shillings a bushel; in consequence of which, he lost £7. 16*s*. by the bargain. How much of each did he buy?

Ans. 48 bushels of the better, and 36 of the worse.

48. Three persons, *A, B,* and *C,* spent equal sums at a tavern. *C* having no money, the reckoning was paid by *A* and *B*. When *C* came to reimburse them, he paid 4 times as much to *A* as to *B*; and observed, that if *B* had paid 3 shillings more of his reckoning, their demands would have been equal. Required the sum each spent, and the respective parts of *C*'s reckoning that *A* and *B* paid.

Ans. Each spent 10 shillings; *A* paid 8, and *B* 2.

49. A certain sum is divided among three persons: A receives £3000 more than the half, B £1000 less than the third part, and C £800 more than the fourth part of the whole. What is the sum divided, and what does each receive?

Ans. The whole is £38400; A receives £16200, B £11800, C £10400.

50. A Farmer had two flocks of sheep, one of which contained 40, and the other was sold for £30; but one sheep of the latter was worth four of the other; and the value of the first flock was only £4 more than the price of eight sheep of the second. How many sheep did the second flock contain, and what was the value of a sheep of each?

Ans. The number was 15, and the prices £2, and 10 shillings.

51. A and B playing at billiards, A bet 5 shillings to 4 on every game, and found that after a certain number of games he had won 10 shillings. Had B won one game more, the number won by him would have been to the number won by A as 3 to 4. How many did each win?

Ans. A won 20, and B 14.

52. A besieged garrison had such a quantity of bread, as would, if distributed to each at 10 ounces a day, last 6 weeks; but having lost 1200 men in a sally, the governor was enabled to increase the allowance to 12 ounces *per* day for 8 weeks. Required the number of men at first in the garrison.

Ans. 3200.

53. A composition of copper and tin containing 100 cubic inches weighed 505 ounces. How many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin to weigh $4\frac{1}{4}$ ounces?

Ans. 420 of copper, and 85 of tin.

54. There are two towns, A and B, which are 131 miles distant from each other. A coach sets out from A at 6 o'clock in the morning, and travels at the rate of 4 miles an hour with-

out intermission, in the direct road towards *B*. At 2 o'clock in the afternoon of the same day a coach sets out from *B* to go to *A*, and goes at the rate of 5 miles an hour constantly. Where will they meet?

Ans. 76 miles from *A*, and 55 from *B*.

55. Out of a certain sum, a man paid his creditors £96; half of the remainder he lent his friend; he then spent one-fifth of what now remained; and after all these deductions had one-tenth of his money left. How much had he at first?

Ans. £128.

56. At the review of an army, the troops were drawn up in a solid mass, 40 deep, when there were just one-fourth as many men in front as there were spectators. Had the depth however been increased by 5, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Of what number of men did the army consist?

Ans. 180000.

57. *A* and *B*, in order to keep up the price of copper to £86 *per* ton, agree for a certain time to sell jointly all the copper they raise; yet so that each shall be paid proportionably to the quantity he raises. Now the whole quantity raised in the stipulated time was 235 tons; and *A* receives £4214 more than *B*. Required the quantity raised by each.

Ans. 142 tons by *A*, and 93 by *B*.

58. Bought two pieces of linen, one of which wanted 12 yards of being four times as long as the other. The longer cost 5 shillings, and the shorter 4 shillings a yard; 23 yards being cut off from the longer, and 5 from the shorter, and the remainders being sold for one shilling a yard more than they cost, I received £7. 2*s*. How many yards of each were there?

Ans. 40, and 13.

59. On the first of January, 1799, a certain beggar received from *A* as many groats as *A* was years old, who repeated a

similar donation every January during the 7 following years, during the last of which *A* died, his alms to the poor man having in all amounted to £7. 18*s*. 8*d*. Determine in what year he was born, and his age at his death.

Ans. He was born in 1743, and had completed 63 years at his death.

60. A Courier, passing through a certain place *A*, travels at the rate of 13 miles in 2 hours; 12 hours afterwards another passes through the same place, travelling the same road, at the rate of 26 miles in 3 hours. How long, and how far must he travel before he overtakes the first?

Ans. 36 hours, and 312 miles.

61. During a panic, there was a run on two Bankers *A* and *B*. *B* stopped payment at the end of three days, in consequence of which the alarm increased, and the daily demand for cash on *A* being trebled, *A* failed at the end of two more days. But if *A* and *B* had joined their capitals, they might both have stood the run, as it was at first, for seven days, at the end of which time *B* would have been indebted to *A* £4000. What was the daily demand for cash on *A*'s Bank at first?

Ans. £2000.

62. As *A* and *B* were going to school, *A* first shot an arrow in the direction in which they were going, which *B* took up and shot forward; and so on alternately till the arrow had passed exactly from one mile-stone to another; when it appeared that *A* had shot the arrow 8 times, and *B* 7 times. Some time afterwards, *A* and *B* were on the opposite banks of a river, the breadth of which they wished to ascertain; *A* first shot the arrow across the river, and it flew 13 yards beyond the bank on which *B* stood; *B* then took it up, and from the place where it had fallen, shot it back across the river; it now fell $9\frac{2}{7}$ yards beyond the bank upon which *A* stood. Required the breadth of the river.

Ans. 100 yards.

63. Three Merchants, A, B, and C, enter into a speculation; B subscribes £10 more than four-fifths of what A does; and C £30 more than half of what B does. A's gain is two-fifths of his subscription, and B's is £148. What are the respective sums subscribed, and whole gain?

Ans. The sums subscribed are 450, 370, and 215 pounds; and the whole gain is £414.

64. A Tenant agreed to pay his Landlord two-thirds of the profit of a farm after deducting the expense of cultivation, and upon this agreement found that his own share was one-sixth of the whole produce. Afterwards the expense of cultivation having fallen in the ratio of 3 : 2, and the value of the produce in the ratio of 5 : 3, he found that adhering to his agreement he must pay his Landlord £400. Determine the original value of the produce.

Ans. £2250.

65. A Gentleman left legacies to his four servants, A, B, C, D, proportional to the time each had been in his service, but in case any of them should die before the expiration of the year, their share to be divided equally among the others. Accordingly B died; and his share so divided made the share of C a mean proportional between those of A and D; whereas before A was to have had £78, C £30, and D £6. What did each receive?

Ans. A received £96, C £48, and D £24.

66. The Colonel of a regiment at Derby sent a detachment to Leeds to protect the Machinery. The detachment was a third of the regiment; but the commanding officer at Leeds thinking his numbers insufficient, and suspecting that 50 of his men had been corrupted by the Luddites, ordered the suspected men back to Derby, with a request that the Colonel would let him have half the regiment. To comply with his wishes, the Colonel had every third man drafted off and sent to Leeds. Of how many men did the regiment consist?

Ans. 900.

67. There are two places, 154 miles distant, from which two persons set out at the same time to meet, one travelling at the rate of 3 miles in two hours, and the other at the rate of 5 miles in four hours. How long, and how far did each travel before they met?

Ans. 56 hours; and 84, and 70 miles.

68. *A* sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and eight hours afterwards *B* sets out from the same place, and travels the same road at the rate of 5 miles in three hours. How long, and how far must *A* travel before he is overtaken by *B*?

Ans. 50 hours, and 70 miles.

69. A Farmer's rent was £50 a year, and his annual expenditure (including the assessed taxes, which amounted to one sixth of his expenses) was such, that he was able to pay his landlord only £30. The year following his rent was lowered 20 *per cent.*; the taxes also were reduced one-half, and agricultural produce increased in value one-third; in consequence he was enabled to pay his rent and former debt, and to lay by £5. What was his expenditure and the value of his produce each year?

Ans. His expenditure was £60 the first year and £55 the second. The value of his produce £90, and £120, respectively.

70. A man lent out a certain sum to interest at £8 *per cent. per* annum. He suffered this to accumulate at simple interest for 12 years; and then putting out the principal and interest at the same rate, found that the present annual interest exceeded the former by £38. 8*s*. Required the sum put out each time.

Ans. £500 the first time, and £980 the second.

71. A Waterman finds by experience that he can with the advantage of a common tide row down a river from *A* to *B*, which is 18 miles, in an hour and a half, and that to return

from B to A against an equal tide, though he rows back along the shore, where the stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter. It is required from hence to find at what rate *per* hour the tide runs in the middle, where it is strongest.

Ans. At the rate of two miles and a half *per* hour.

72. The ingredients of a loaf of bread are rice, flour, and water, and the weight of the whole is 15lbs. The weight of the rice augmented by 5lbs. is two-thirds of the weight of the flour, and the weight of the water is one-fifth of the weight of the flour and rice together. Required the weight of each.

Ans. Rice 2lbs., flour $10\frac{1}{2}$lbs., water $2\frac{1}{2}$lbs.

73. In a battery two cannon were employed, the first of which had been fired 36 times before the second began to play; and afterwards was fired eight times whilst the second was fired seven. But the quantity of powder used for each shot of the first was less than what was used for the second in the proportion of 3 : 4. How many times was the second fired, before it had consumed as much powder as the first?

Ans. 189 times.

74. Suppose two fingers of a watch (a) and (b) were together on Sunday noon at 12 o'clock, and that the motion of each was such that (a) moved round the horary circle in one hour, and (b) in $1\frac{1}{60}$ hour. When will they be together again for the first time?

Ans. 61 hours.

75. A Draper bought a piece of cloth for £69, from which he cut off 11 yards. He then met with another piece of equal goodness, for which he gave £21, and found that if it had been one yard longer, its length would have been to the length of the remainder of the first as 2 to 3. How many yards were there in each piece, and what was the price of a yard?

Ans. 23 in the first, and 7 yards in the second; and the price £3 *per* yard.

76. A Fruiterer sells for 19*s*. 6*d*. a certain number of oranges and apples, of which the latter exceeded the former by 180. He sells the apples at the rate of five for 3*d*., and fifteen oranges bring him in $1\frac{1}{2}d$. more than 35 apples. How many are there of each sort, and what are the oranges worth apiece?

Ans. 240 apples, and 60 oranges, which are worth $1\frac{1}{4}d$. each.

77. Divide the number 198 into five such parts, that the first increased by one, the second increased by two, the third diminished by three, the fourth multiplied by four, and the fifth divided by 5, may be all equal.

Ans. 23, 22, 27, 6, and 120, are the numbers.

78. A person has four casks, the second of which being filled from the first, leaves the first four-sevenths full. The third being filled from the second leaves it one-fourth full; and when the third is emptied into the fourth, it is found to fill only nine-sixteenths of it. But the first will fill the third and fourth, and have fifteen quarts remaining. How many quarts does each hold?

Ans. 140, 60, 45, and 80 respectively.

79. A packet sailing from Dover with a fair wind, arrives at Calais in two hours; and on its return the wind being contrary, it proceeds six miles an hour slower than it went. Now when it is half way over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done had the wind not changed, in the proportion of 6 : 7. Required the rates of sailing and the distance between Dover and Calais.

Ans. The distance is 22 miles, and in returning it sails 5 and 7 miles an hour.

80. Six hundred persons voted upon a disputed question, which was lost by a certain number. The same number of persons having voted again upon the same question, it was from some change in circumstances carried by twice as many as

it was before lost by; and the new majority was to the former one as 8 : 7. How many changed their minds?

Ans. 150.

81. The Gas Contractors engage to light a shop with 5 large and 3 small burners; but having by them only one large burner, supply the deficiency with five small ones. The shopkeeper not finding this light sufficient, procures 2 more small burners, and at the same time agrees for the light to burn double the usual time on Saturday nights, for which additional gas he was required to pay £1. 11*s.* How much did he pay altogether?

Ans. 5 guineas.

82. *A* and *B* set out from two places, *C* and *D*, at the same time, towards *E*; the road from *C* to *E* being through *D*. *A* travels 7 miles an hour, and at that rate of travelling would have overtaken *B* 5 miles before he got to *E*; but after arriving at *D*, he travels $6\frac{2}{3}$ miles an hour, in consequence of which he overtakes *B* just as he enters *E*. Supposing *B* to travel 5 miles an hour, what are the distances between *C*, *D*, and *E*?

Ans. From *C* to *D* 14 miles, and from *D* to *E* 40.

83. A Gentleman wishing his two daughters to receive equal portions when they became of age, bequeathed to the elder the accumulated interest of a certain sum of money, bought at the time of his death into the 4 *per cent.* stock at 88; and to the younger the accumulated interest of a sum less than the former by £3500, bought at the same time into the 3 *per cents.* at 63. Supposing their ages at the time of their father's death to have been 17 and 14, what would be the sum bought into the stocks in each case, and what would be the fortune of each?

Ans. The sums would be £7700, and £4200; and fortune £1400.

84. Two persons, *A* and *B*, start at the same time for a race which lasted six minutes. Now after galloping four minutes at

the same uniform pace at which each started, the distance between them is $\frac{1}{110}$th part of the whole length of the course. They continue to run for one minute more at the same speed as at first; and then *B*, who is last, quickens the speed of his horse 20 yards a minute, and comes in exactly two yards before *A*, whose horse had run at the same uniform pace throughout. What was the length of the course?

Ans. 3 miles.

85. Out of a common pack of cards, a certain number, including the ten of diamonds, was dealt equally amongst four persons, the dealer turning up the last card, which was the ten of spades, which he gave himself. Now if twice the number of cards had been dealt to each, the ten of spades being turned up by the dealer, and the ten of diamonds being still dealt out, the chance of the dealer's having the ten of diamonds would be to the chance against him as 3 : 10. Required the number of cards dealt to each the second time.

Ans. 10.

86. Two companies of soldiers, consisting of equal numbers, were sent out under *A* and *B*, from two hostile camps, to reconnoitre. Falling in with each other, a skirmish ensued, in which *A* lost 50 killed and prisoners, and *B* had 20 killed. *A* however having been reinforced by a party equal to five-sevenths of the number which *B* had remaining, and *B* having been reinforced by a number greater by 46 than three-fifths of the number which *A* had remaining, they renewed the engagement, when *A* was forced to retire with the additional loss of 30 men. When the returns were made, *B* found he had again lost 20 men, but that he had then twice as many men remaining as *A* had. How many had each at first?

Ans. 90.

87. A Farm was rated at 3*s.* an acre; and the Tenant on receiving back at his rent-day 10 *per cent.* of his rent, found that the sum returned amounted to £6 more than the whole rate. The next year the rates were doubled, and he received

back 15 *per cent.* of his rent; but he now found that the sum returned only just paid for the whole rate. What was the rent of the farm, and of how many acres did it consist?

Ans. The rent was £240; and there were 120 acres.

88. In a Tithe Commutation the rent-charge was fixed at 3*s.* an acre; and the Tithe Owner found the first year that his rates wanted £6 of being 10 *per cent.* on his receipts. The next year the rates were doubled, and amounted to 15 *per cent.* on his receipts. What was the number of acres in the parish?

Ans. 1600.

89. A Sportsman, who kept an account of the number of birds which he killed, found that each succeeding season he wanted 50, in order that the number killed might bear the proportion of 3 : 2 to the number killed in the preceding year. In the fourth year he found that he had killed 170 fewer than three times the number killed in the first year. How many did he kill the first year?

Ans. 180.

90. Several detachments of artillery divided a certain number of cannon balls. The first took 72, and one-ninth of the remainder; the next 144, and one-ninth of the remainder; the third 216, and one-ninth of the remainder; the fourth 288, and one-ninth of those that were left; and so on; when it was found that the balls had been equally divided. Determine the number of detachments and balls.

Ans. 4608 balls, and 8 detachments.

91. *A* entered into a Canal speculation with 14 others, and the profits of this concern amounted in all to £595 more than five times the price of an original share. Seven of his former partners in this affair joined with him in a scheme for navigating the said Canals with steam-boats, each venturing a sum of money less than his former gains by £173. But the steam-boats unexpectedly blowing up, *A* found he had lost £419 by them, for the company not only never recovered the money ad-

vanced, but had lost all they had gained by digging the Canals and £368 besides. What were the prices of shares in the two concerns originally?

Ans. £700 in the first speculation, and £100 in the second.

92. A Merchant wishing to buy a certain quantity of pimento, the price of which he calculates at the rate of 5 bags for £8, transmits to his foreign agent the requisite sum of money. Before the order arrives, pimento has risen in value; and the money is sufficient only to buy a quantity less by 18 bags than that which the Merchant intended. It appears also that as many bags as exceed one-third of the intended quantity by $5\frac{1}{4}$, will now cost £10. 7*s*. more than they would have done, had the price not varied. What is the quantity purchased?

Ans. 432 bags.

93. Four men walking abroad found a purse containing shillings only, out of which every one of them took a number at a venture. Afterwards comparing their numbers together, they found that if the first took 25 shillings from the second, it would make his number equal to what the second had left. If the second took 30 shillings from the third, his money would then be triple what the third had left. And if the third took 40 shillings from the fourth, his money would then be double of what the fourth had left. Lastly, the fourth taking 50 shillings from the first, he would then have three times as much as the first had left, and five shillings over. What had each?

Ans. 100, 150, 90, and 105 shillings, respectively.

94. Fifteen current guineas should weigh 4 ounces; but a parcel of light gold being weighed and counted, was found to contain 9 more guineas, than was supposed from the weight; and a part of the whole, exceeding the half by 10 guineas and a half, was found to be $1\frac{1}{2}$ oz. deficient in weight. What was the number of guineas?

Ans. 189.

95. A Merchant bought a quantity of wheat for £200, half of which he reserved for his private use. He then sold 5 bushels more than $\frac{3}{4}$ of the remaining quantity at such a price as to gain £40 *per* cent. But the price of wheat having advanced, he sold the remainder at such a price as to gain £67 *per* cent. by what he sold. And had the whole been sold at this latter price, he would have gained £160 *per* cent. How much did he buy, and how did he sell it?

Ans. He bought 400 bushels; and sold the first portion at 14*s.*, and the second at 26*s.* *per* bushel.

96. A Brewer, from a certain quantity of ingredients which cost £20, brews 500 gallons of ale (on which there is a duty of 6*d.* a gallon), and sells it at 2*s.* a gallon. Afterwards, from the same quantity of ingredients, he brews a certain number of gallons of strong beer (on which he pays the ale duty), and the remainder small beer, making together the same number of gallons as before;—when by mixing them together, and selling the mixture as ale, he finds his gains increased in the proportion of 10 : 7. Determine the number of gallons of strong beer, supposing the duty on small beer one-fourth of that on ale.

Ans. 100 gallons.

VII. *Problems producing Simple Equations, involving two unknown Quantities.*

1. A Draper bought two pieces of cloth for £12. 13*s.*; one being 8*s.*, and the other 9*s.* *per* yard. He sold them each at an advanced price of 2*s.* *per* yard, and gained by the whole £3. What were the lengths of the pieces?

Ans. 17 yards the first, and 13 the second.

2. A bill of £26. 5*s.* was paid with half guineas and crowns, and twice the number of half guineas exceeded three times the number of crowns by 17. How many were there of each?

Ans. 40 half guineas, and 21 crowns.

3. Two labourers, *A* and *B*, received £5. 17*s*. for their wages; *A* having been employed 15, and *B* 14 days; and *A* received for working four days 11*s*. more than *B* did for three days. What were their daily wages?

Ans. *A* had 5*s*., and *B* 3*s*. a day.

4. A person had two casks, the larger of which he filled with ale, and the smaller with cyder. Ale being half a crown, and cyder 11*s*. *per* gallon, he paid £8. 6*s*.; but had he filled the larger with cyder, and the smaller with ale, he would have paid £11. 5*s*. 6*d*. How many gallons did each hold?

Ans. The larger contained 18, and the smaller 11 gallons.

5. A person expends half a crown in apples and pears, buying his apples at 4, and his pears at 5 a penny; and afterwards accommodates his neighbour with half his apples and one-third of his pears for 13 pence. How many did he buy of each?

Ans. 72 apples, and 60 pears.

6. Two persons, *A* and *B*, played cards, each with a different sum. After a certain number of games, *A* had won half as much as he had at first, and found that if he had 15*s*. more, he would have had just three times as much as *B*. But *B* afterwards won 10*s*. back, and he had then twice as much as *A*. What had each at first?

Ans. *A* had 14, and *B* 19 shillings.

7. A certain sum of money put out to interest, amounts in 8 months to £297. 12*s*.; and in 15 months its amount is £306 at simple interest. What is the sum, and the rate *per cent.?*

Ans. £288, at 5 *per cent.*

8. A Farmer being asked how many quarters of wheat he had sold in the market, answered, if he had sold 8 quarters more, and got 7*s*. *per* quarter more than he did, he should have received £11. 15*s*. more than he had: but if he had sold 7 quarters more at 8*s*. *per* quarter more, he should have had

£11. 17*s.* more. How many quarters did he sell, and what was the price?

Ans. 13 quarters, at 11*s. per* quarter.

9. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of its digits, the quotient is 4; but if the digits be inverted, and that number divided by a number greater by 2 than the difference of the digits, the quotient becomes 14. Required the number.

Ans. 48.

10. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Ans. $\frac{4}{5}$.

11. A Farmer parting with his stock, sells to one person 9 horses and 7 cows for £300; and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each?

Ans. The price of a cow was £12, and of a horse £24.

12. A Farmer hires a farm for £245 *per ann.*, the arable land being valued at £2 an acre, and the pasture at 28 shillings; now the number of acres of arable is to half the excess of the arable above the pasture as 28 : 9. How many acres were there of each?

Ans. 98 acres of arable, and 35 of pasture.

13. A person owes a certain sum to two creditors. At one time he pays them £53, giving to one four-elevenths of the sum which is due, and to the other £3 more than one-sixth of his debt to him. At a second time he pays them £42, giving to the

first three-sevenths of what remains due to him, and to the other one-third of what is due to him. What were the debts?

Ans. £121, and £36.

14. A and B playing at backgammon, A bet 3*s.* to 2*s.* on every game, and after a certain number of games found that he had lost 17 shillings. Now had A won 3 more from B, the number he would then have won, would have been to the number B would have won as 5 to 4. How many games did they play?

Ans. 9.

15. A Vintner has 2 casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining in the proportion of 8 to 3. After they become half empty, he puts 10 gallons of water into each, and finds that the quantities of liquor now in them are as 9 to 5. How many gallons will each hold?

Ans. The larger 79, and the smaller 35 gallons.

16. A person having laid out a rectangular bowling-green, observed that if each side had been 4 yards longer, the adjacent sides would have been in the ratio of 5 to 4; but if each had been 4 yards shorter, the ratio would have been 4 to 3. What are the lengths of the sides?

Ans. 36, and 28 yards.

17. At an election for two members of parliament, three men offer themselves as candidates, and all the electors give single votes. The numbers of voters for the two successful ones are in the ratio of 9 to 8; and if the first had had 7 more, his majority over the second would have been to the majority of the second over the third as 12 : 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 7. How many voted for each?

Ans. 369, 328, and 300, respectively.

18. Determine three numbers, such that if 6 be added to the first and second, the sums will be in the proportion of 2 : 3; if 5 be added to the first and third, the sums will be in the proportion of 7 : 11; but if 36 be subtracted from the second and third, the remainders will be as 6 : 7.

Ans. 30, 48, 50.

19. Two shepherds, *A* and *B*, are intrusted with the charge of two flocks of sheep. *A*'s consisting chiefly of ewes, many of which produced lambs, is at the end of the year increased by 80; but *B* finds his stock diminished by 20; when their numbers are in the proportion of 8 to 3. Now had *A* lost 20 of his sheep, and *B* had an increase of 90, the numbers would have been in the proportion of 7 to 10. What were the numbers?

Ans. *A*'s 160, and *B*'s 110.

20. Two persons, *A* and *B*, can perform a piece of work in 16 days. They work together for 4 days, when *A* being called off, *B* is left to finish it, which he does in 36 days more. In what time would each do it separately?

Ans. *A* in 24 days, and *B* in 48 days.

21. There is a cistern, into which water is admitted by three cocks, two of which are exactly of the same dimensions. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

Ans. Each of the equal ones in 32 hours, and the other in 24.

22. Some hours after a courier had been sent from *A* to *B*, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered *B*; in order to which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles is 13 hours and 40 minutes. How many miles does each go *per* hour?

Ans. The first goes 3, and the second 7 miles an hour.

23. Two loaded waggons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7, being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then 10 tons. What were the weights at first?

Ans. 16, and 20 tons.

24. A Gentleman gave away a certain sum in charity to 14 men and 15 women. Had the sum been less by 12 shillings and only half the number of men relieved, the rest being divided amongst the women, each woman would have received two shillings more than each man did. But if there had been only 8 women, and the rest had been divided amongst the men, each man would have received twice as much as each woman. How much money was given away?

Ans. 24 guineas.

25. When wheat was 5 shillings a bushel, and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6 shillings left. How must he lay out his money, and fill his sack?

Ans. He must buy 9 bushels of wheat, and 12 bushels of rye.

26. A Stage Coach carries six inside, the fare outside is 13*s.*, and one-third of the sum of the outside fares exceeds one-fifth of those inside by £1. 1*s.* 8$\frac{1}{4}$*d.* An opposition arising, the coachman loses three outside and two inside passengers, and also reduces the inside fare by 5*s.* and halves the outside; and then the whole loss is £7. 0*s.* 6*d.* Find the number of outside places, and the fare inside.

Ans. 10 outside places, and 18*s.* fare inside.

27. A Draper bought two pieces of cloth of different kinds for £37. 4*s.*: there were 6 yards of the coarser more than there

were of the finer; and had the coarser cost 2 shillings a yard more than it did, 6 yards of the coarser would have cost just as much as 5 yards of the finer. He afterwards bought 4 yards of the finer, and 12 of the coarser at the same prices *per* yard, and found their value less than that of the former pieces in the ratio of 20 : 31. How many yards did he buy the first time, and what did he give *per* yard for each?

Ans. 9 yards of the finer, and 15 of the coarser; and the prices were 36, and 28 shillings *per* yard.

28. A Mercer bought two pieces of silk of different lengths for £50; the price of two yards of the shorter was 6*s.* 8*d.* more than the price of 3 yards of the longer; and each piece cost the same sum. He cut off two yards from each, and sold the rest for £53. 12*s.* Now if he had sold the whole at that rate, he would have gained £5 by each piece. How many yards did each piece contain?

Ans. 25, and 15 yards.

29. *A* sets out express from *C* towards *D*, and three hours afterwards *B* sets out from *D* towards *C*, travelling 2 miles an hour more than *A*. When they meet, it appears that the distances they have travelled are in the proportion of 13 to 15; but had *A* travelled five hours less, and *B* gone 2 miles an hour more, they would have been in the proportion of 2 : 5. How many miles did each go *per* hour, and how many hours did they travel before they met?

Ans. *A* went 4, and *B* 6 miles an hour, and they travelled 10 hours after *B* set out.

30. An Omnibus starts with a certain number of passengers, and takes up four more on the road, whose fare is the same as that paid by the others. On deducting one-twelfth of the whole fare for expenses, there remains a gain of 4*s.* 7*d.* But if those who were taken in last had paid half as many pence as there were passengers altogether, the money received would have exceeded double the difference of the sum actually paid by

2*s.* 8*d.* With how many did the omnibus start, and what was the fare of each?

Ans. It started with 6 passengers, and fare was 6*d.*

31. The revenue of a state was increased to provide for a war in the ratio of $2\frac{1}{4} : 1$; and after deducting the expense of collecting, and the interest of the National Debt, the available income was augmented in the ratio of $3\frac{12}{23} : 1$. Now it was found upon calculation, that had circumstances on the contrary permitted the revenue to be reduced in the ratio of $1\frac{7}{9} : 1$, the sum remaining after the specified deductions would have been diminished in the ratio of $7\frac{2}{3} : 1$, and would in fact have only amounted to four millions. Required the amount of the revenue, and the interest of the debt; on supposition that the expense of collecting varies as the square root of the amount collected.

Ans. The revenue before the increase was 64 millions, and the interest of the National Debt 28 millions.

32. *A* and *B* engaged to reap a field of corn in 12 days. The times in which they could severally reap an acre are as 2 : 3. After some time, finding themselves unable to finish it in the stipulated time, they called in *C* to help them; whose rate of working was such, that if he had wrought with them from the beginning, it would have been finished in 9 days. Also the times in which he could severally have reaped the field with *A* alone, and with *B* alone, are in the proportion of 7 to 8. When was *C* called in?

Ans. After 6 days.

33. Two mixtures are made of brandy and sherry; the quantities of brandy in each being as 4 to 3; and the difference of the quantities of sherry being greater by 25 gallons than the difference of the quantities of brandy. Also if three times the quantity of brandy had been put into the first mixture, and twice the quantity into the second, the quantities of brandy would have been proportional to the quantities of sherry. But if the sherry in the second mixture had been mixed with the

brandy in the first, and the sherry in the first with the brandy in the second, the whole mixtures would then have been in the ratio of 5 to 6. Required the quantities of brandy and sherry in each mixture.

Ans. The quantities of brandy are 80, and 60 gallons, and the quantities of sherry are 90, and 45 gallons.

34. Fine Gold Chains are manufactured at Venice, and are sold at so much *per* braccio, a measure containing about two feet English. When there are 90 links in an inch, the value of the workmanship of a braccio is equal to the whole value of a braccio when there are but 30 links in an inch; and the whole value of the braccio in the former case is equal to three times the difference between the cost of the material and workmanship of a braccio in the latter, together with $4\frac{4}{9}$ francs. Supposing that the workmanship in each braccio varies as the number of links in an inch, and the weight of the metal inversely as the square of that number; find the values of the material and workmanship in a braccio of each of the chains.

Ans. The values of the materials are $4\frac{4}{9}$, and 40 francs, respectively; and of the workmanship, 60, and 20.

35. During a winter, when fuel was scarce, two men, *A* and *B*, went in quest of coals and turf, which they agreed to use in common. *A* met with three bushels of coals, and *B* two, at the same price *per* bushel, and also seven baskets of turf. *A* stipulated that he should consume twice as many coals as *B*. *B* assented, but demanded of him 2*s*. 10*d*. When this stock was exhausted, *B* purchased one bushel of coals, and *A* five, together with 6 baskets of turf, at the same rates respectively as before; but now *B* consumed three times as many coals as *A*, and paid him 18*s*. 6*d*. What was the price of a bushel of coals, and of a basket of turf; equal quantities of turf having been consumed by each person?

Ans. The price of a bushel of coals was 5*s*., and of a basket of turf 4*d*.

36. Two Spanish muleteers, *A* and *B*, were seated under a tree in order to dine; and on examining, found their stock

of provisions to consist of 5 small loaves of bread, three of which were *A*'s property, and a bottle of wine, which was *B*'s. A stranger, who happened to come up at the time, was invited to partake of their fare, which was just sufficient for three persons; and at parting, being pleased with their behaviour, he gave them what Spanish money he had about him, which amounted to 6*s*. $5\frac{1}{4}$*d*., to be equitably shared between them. Now as many shillings as a loaf cost pence would, with four pence more, at the next town have bought six such loaves, and four bottles of the same wine; and when the money was divided, *B* received 1*s*. $10\frac{1}{2}$*d*. more than *A*. What was the price of each loaf, and a bottle of wine?

Ans. A loaf cost 7 pence, and a bottle of wine $11\frac{1}{2}$ pence.

VIII. *Problems producing Pure Equations.*

1. FIND two numbers, which are in the proportion of 8 to 5, and whose product is equal to 360.

Ans. $\pm$ 24, and $\pm$ 15.

2. There are two numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers?

Ans. $\pm$ 18, and $\pm$ 14.

3. In a court there are two square grass-plots; a side of one of which is 10 yards longer than the side of the other; and their areas are as 25 to 9. What are the lengths of the sides?

Ans. 25, and 15 yards.

4. A person bought two pieces of linen, which together measured 36 yards. Each of them cost as many shillings *per* yard, as there were yards in the piece; and their whole prices were in the proportion of 4 to 1. What were the lengths of the pieces?

Ans. 24, and 12 yards.

5. There are two numbers, whose sum is to the less as 5 to 2; and whose difference, multiplied by the difference of their squares, is 135. Required the numbers.

Ans. 9, and 6.

6. There are two numbers, which are in the proportion of 3 to 2; the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Required the numbers.

Ans. 6, and 4.

7. There is a field in the form of a rectangular parallelogram, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to one-sixth of the whole, being planted, there remain for ploughing 625 square yards. What are the dimensions of the field?

Ans. The sides are 30, and 25 yards.

8. Some Gentlemen made an excursion; and every one took the same sum. Each gentleman had as many servants attending him as there were gentlemen; and the number of pounds which each had was double the number of all the servants; and the whole sum of money taken out was £3456. How many gentlemen were there?

Ans. 12.

9. Divide the number 49 into two such parts, that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$.

Ans. 28, and 21.

10. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the regiment; but these being found to be insufficient, each company furnished 3 more men; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment?

Ans. 12.

11. A charitable person distributed a certain sum amongst some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received one-third of as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. Now the men received all together 18*s*. more than the women. How many were there of each?

Ans. 12 men, and 15 women.

12. A Gentleman who had a certain number of horses, kept part of them at livery stables, for which he paid £4. 10*s*. *per* week. The rest he kept at home, and their number was to the number kept at the livery stables as 7 to 3. He found that the expense of keeping 5 at home was just equal to that of keeping 4 at the stables; and the number of shillings that one horse cost him at home was to the number of horses kept at home as 6 to 7. How many horses had he?

Ans. 6 at the livery stables, and 14 at home.

13. A city barge, with chairs for the company and benches for the rowers, went a summer excursion, with two bargemen on every bench. The number of gentlemen on board was equal to the square of the number of bargemen, and the number of ladies was equal to the number of gentlemen, twice the number of bargemen, and one over. Among other provisions, there were a number of turtles equal to the square root of the number of ladies; and a number of bottles of wine less than the cube of the number of turtles by 361. The turtles in dressing consumed a great quantity of wine, and the party having stayed out till the turtles were all eaten, and the wine all gone, it was computed, that supposing them all to have consumed an equal quantity, (viz. gentlemen, ladies, bargemen, and turtles,) each individual would have consumed as many bottles as there were benches in the barge. Required the number of turtles.

Ans. 19.

14. From two towns, *C* and *D*, two travellers, *A* and *B*, set out to meet each other; and it appeared that when they met,

B had gone 35 miles more than three-fifths of the distance that *A* had travelled; but from their rate of travelling, *A* expected to reach *C* in 20 hours and 50 minutes; and *B* to reach *D* in 30 hours. Required the distance of *C* from *D*.

Ans. 275 miles.

15. A Farmer bought two flocks of sheep, the first of which contained 18 fewer than the second. If he had given for the first flock as many pounds as there were sheep in the second, and for the second as many pounds as there were sheep in the first, then the price of 6 sheep of the first flock would have been to the price of 7 sheep of the second in the proportion of 7 to 6. Required the numbers in each flock.

Ans. 108, and 126.

16. A Poulterer bought a number of ducks and turkeys, the number of ducks exceeding the number of turkeys by 8. For each duck he gave half as many shillings as there were turkeys, and for each turkey half as many shillings as there were ducks. He afterwards bought another small flock of turkeys, containing 4 fewer than the number of turkeys he bought before; and having given for each of them as many shillings as there were turkeys in the flock, he found, that if his former purchase had cost 16 shillings more, it would have cost exactly four times as much as the present one. How many ducks and turkeys did he buy at first?

Ans. 12 turkeys, and 20 ducks.

17. Two men, *A* and *B*, entered into partnership with stocks, which are in the proportion of 9 to 8; and after trading one year, *A* found his share of their gain to amount to one-third of his stock. They continued to trade for as many years as are equal to three-fourths of the number of pounds which *B* contributed to the stock, and found their whole gain amount to £1666. What did each contribute to the stock; and how many years did they trade?

Ans. *A* contributed £63, and *B* £56; and the number of years is 42.

18. A person wishing to ascertain the area of a certain quadrilateral field, found that he could determine it the most readily by dividing it into two portions, one of which was of the form of a rectangular parallelogram, the shorter side of which measured 60 yards. The other was of the form of a right-angled triangle, whose shortest side was equal to the shorter side of the parallelogram, and the other side containing the right angle, was equal to the diagonal of the parallelogram; and the area of the triangle was to the area of the parallelogram as 5 to 8. What was the area of the field?

Ans. 7800 square yards.

19. A Merchant laid out a certain sum upon a speculation, and found at the end of a year that he had gained £69. This he added to his stock, and at the end of another year found that he had gained exactly as much *per* cent. as in the year preceding. Proceeding in the same manner, and each year adding to his stock the gain of the year preceding, he found at the beginning of the fifth year that his stock was to the original stock as 81 to 16. What was the sum he first laid out?

Ans. £138.

20. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number.

Ans. 23.

21. From two towns, C and D, which were at the distance of 396 miles, two persons, A and B, set out at the same time, and met each other, after travelling as many days as are equal to the difference of the number of miles they travelled *per* day; when it appears that A has travelled 216 miles. How many miles did each travel *per* day?

Ans. A went 36, and B 30.

22. There are two numbers, whose sum is to the greater as 40 is to the less, and whose sum is to the less as 90 is to the greater. What are the numbers?

Ans. 36, and 24.

23. It is required to find two numbers such, that the product of the greater and the cube of the less may be to the product of the less and the cube of the greater as 4 to 9; and the sum of the cubes of the numbers may be 35.

Ans. 3, and 2.

24. The paving of two square court-yards cost £205; a yard of each costing one-fourth of as many shillings as there were yards in a side of the other. And a side of the greater and less together measure 41 yards. Required the length of a side of each.

Ans. 25, and 16 yards.

25. A person bought a number of apples and pears, amounting together to 80. Now the apples cost twice as much as the pears: but had he bought as many apples as he did pears, and as many pears as he did apples, his apples would have cost 10*d.*, and his pears 3*s.* 9*d.* How many did he buy of each?

Ans. 60 apples, and 20 pears.

26. A person exchanged a quantity of brandy for a quantity of rum and £11. 5*s.*; the brandy and rum being each valued at as many shillings *per* gallon as there were gallons of that liquor. Now had the rum been worth as many shillings *per* gallon as the brandy was, the whole value of the rum and brandy would have been £56. 5*s.* How many gallons were there of each?

Ans. 25 gallons of brandy, and 20 of rum.

27. There are two rectangular vats, the greater of which contains 20 solid feet more than the other. Their capacities are in the ratio of 4 to 5; and their bases are squares, a side of

each of which is equal to the depth of the other. What are the depths?

Ans. 5 feet, and 4 feet.

28. Bought two square carpets for £62. 1*s.*; for each of which I paid as many shillings *per* yard as there were yards in its side. Now had each of them cost as many shillings *per* yard as there were yards in the side of the other, I should have paid 17*s.* less. What was the size of each?

Ans. One contained 81, and the other 64 square yards.

29. The number of men in both fronts of two columns of troops, *A* and *B*, when each consisted of as many ranks as it had men in front, was 84; but when the columns changed ground, and *A* was drawn up with the front *B* had, and *B* with the front *A* had, the number of ranks in both columns was 91. Required the number of men in each column.

Ans. 2304, and 1296.

30. A field in the form of a rectangular parallelogram was planted with trees placed at such distances as to have four on every square yard. The expense of planting was such, that every 40 trees cost one-third of as many shillings as there were yards in the diagonal of the parallelogram. But had they been planted at such a price as that every hundred should have cost as many shillings as there were yards in the shorter side of the parallelogram, the expense would have been less by £224. Now a square described upon the diagonal of the parallelogram would be equal to eight-thirds of the square described on the less side, together with the square described on a line which is equal to the difference of the sides. Required the dimensions of the parallelogram.

Ans. The longer side is 80, and the shorter 60 yards.

31. *A* and *B* are two towns, situated on the bank of a river, which runs at the rate of 4 miles an hour. A Waterman rows from *A* to *B*, and back again, and finds that he is 39 minutes longer upon the water than he would have been, had there

been no stream. The next day he repeats his voyage with another waterman, with whose assistance he can row half as fast again; and they find that they are only eight minutes longer in performing their voyage than they would have been, had there been no stream. Determine the rate at which the waterman would row by himself.

Ans. 6 miles *per* hour.

32. There are three towns, *A*, *B*, *C*, the straight lines joining which form a right-angled triangle; *B* being situated at the right angle, and the distance from *A* to *B* being the least of the three. A pedestrian making a circuit of them, at an uniform rate, finds that the time of his going from *A* to *B*, together with the time of going from *B* to *C*, exceeds the time from *C* to *A* by two hours and forty minutes. A coach, which left *A*, to make the same circuit, four hours after the pedestrian, overtakes him at the end of the eighth mile from *B* to *C*; the rate of the coach's travelling being three times that of the pedestrian; and after reaching *A*, and waiting there six hours and forty minutes, it sets out again to make the same circuit, and arrives again at *A* exactly at the same time with the pedestrian, who had rested four hours at *C*. Find the distances of the towns from each other, and the rates of travelling of the pedestrian and the coach.

Ans. The distances are 10, 24, and 26 miles respectively; and the rates of travelling of the pedestrian and the coach are 3 and 9 miles *per* hour.

IX. *Problems producing Adfected Quadratics.*

1. WHAT two numbers are those, whose sum is 19, and whose difference multiplied by the greater is 60?

Ans. 12, and 7.

2. If the square of a certain number be taken from 40, and the square root of this difference be increased by 10, and the

sum multiplied by 2, and the product divided by the number itself, the quotient will be 4. Required the number.

Ans. 6.

3. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 16 yards; and it contains 960 square yards. Required the length and breadth.

Ans. 40, and 24 yards.

4. A person being asked his age, answered, If you add the square root of it to half of it, and subtract 12, there will remain nothing. Required his age.

Ans. 16.

5. Two casks of ale were bought for £2. 18*s.*, one of which contained 5 gallons more than the other, and the price *per* gallon was 2 shillings less than one-third of the number of gallons in the less. Required the number of gallons in each, and the price *per* gallon.

Ans. The numbers were 12, and 17, and the price *per* gallon 2 shillings.

6. From two places, at the distance of 320 miles, two persons, *A* and *B*, set out at the same time to meet each other. *A* travelled 8 miles a day more than *B*, and the number of days in which they met was equal to half the number of miles *B* went in a day. How many miles did each travel *per* day, and how far did each travel?

Ans. *A* went 24, and *B* 16 miles *per* day; *A* went 192, and *B* 128 miles.

7. The difference between the hypothenuse and base of a right-angled triangle is $= 6$, and the difference between the hypothenuse and the perpendicular is $= 3$. What are the sides?

Ans. 15, 12, and 9.

8. In a parcel which contains 24 coins of silver and copper, each silver coin is worth as many pence as there are copper

coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each?

Ans. 6 of one, and 18 of the other.

9. A Farmer received £7. 4*s.* for a certain quantity of wheat, and an equal sum at a price less by 1*s.* 6*d.* *per* bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there of each?

Ans. 32 bushels of wheat, and 48 of barley.

10. Two messengers, *A* and *B*, were dispatched at the same time to a place 90 miles distant; the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before him. At what rate did each travel *per* hour?

Ans. *A* went 10, and *B* 9 miles *per* hour.

11. Bought a number of books, consisting of folios, quartos, and octavos, for £96. 12*s.* Fourteen folios (which was the whole number) cost 3 times as much as all the quartos; and one quarto cost as many shillings as there were quartos. The number of octavos was 32, and their value was such, that 4 of them cost as much as one quarto. Required the value of each, and the number of quartos.

Ans. There were 21 quartos, each folio cost 4½ guineas, each quarto one guinea, and each octavo 5*s.* 3*d.*

12. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the same journey. How many miles did he go *per* hour?

Ans. 7 miles.

13. Bought two flocks of sheep for £65. 13*s.*, one containing 5 more than the other. Each sheep cost as many shillings as there were sheep in the flock. Required the numbers in each flock.

Ans. 23, and 28.

14. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares?

Ans. 21, and 25.

15. What number is that, to which if 24 be added, and the square root of the sum extracted, this root shall be less than the original quantity by 18?

Ans. 25.

16. After taking the kings, queens, and knaves out of a pack of cards, the rest were divided into three heaps. The number of pips contained in the second heap was found to be 4 times the square of the number in the first heap; and had the third heap contained 5 more pips than it did, the number in it would have been exactly half of what the first and second heap contained. Required the number of pips in each heap.

Ans. 6, 144, and 70.

17. A Tailor bought a piece of cloth for £147, from which he cut off 12 yards for his own use, and sold the remainder for £120. 5*s*., gaining 5 shillings *per* yard. How many yards were there, and what did it cost him *per* yard?

Ans. 49 yards, at £3 *per* yard.

18. A regiment of foot was ordered to send 216 men on garrison duty, each company being to furnish an equal number; but before the detachment marched, 3 of the companies were sent on another service, when it was found that each company that remained was obliged to furnish 12 additional men, in order to make up the complement 216. How many companies were there in the regiment, and what number of men was each company ordered to send at first?

Ans. There were 9 companies; and each was to send 24 men.

19. A Poulterer bought 15 ducks and 12 turkeys for five guineas. He had two ducks more for 18 shillings than he had of turkeys for 20 shillings. What was the price of each?

Ans. The price of a duck was 3*s.* and of a turkey 5*s.*

20. Two men, *A* and *B*, entered into a speculation, to which *B* subscribed £15 more than *A*. After 4 months, *C* was admitted, who added £50 to the stock; and at the end of 12 months from *C*'s admission, they found they had gained £159; when *A* withdrawing received for principal and gain £88. What did he originally subscribe?

Ans. £40.

21. A wall was built round a rectangular court to a certain height. Now the length of one side of the court was two yards less than 8 times the height of the wall, and the length of the adjacent side was 5 yards less than 6 times the height of the wall; and the number of square yards in the court was greater than the number in the wall by 178. Required the dimensions of the court, and the height of the wall.

Ans. The sides were 30, and 19, and the height 4 yards.

22. A ship containing 74 sailors, and a certain number of soldiers, besides officers, took a prize. The sailors received each one-third as many pounds as there were soldiers, and the soldiers received £3 a piece less, and £768 fell to the share of the officers. Had the officers however received nothing, the soldiers and sailors might have received half as many pounds *per* man, as there were soldiers. How many soldiers were there, and how much did each receive?

Ans. There were 36 soldiers, each soldier received £9, and each sailor £12.

23. A Poulterer going to market to buy turkeys, met with four flocks. In the second were 6 more than three times the square root of double the number in the first. The third contained three times as many as the first and second; and the fourth contained 6 more than the square of one-third of the

number in the third; and the whole number was 1938. How many were there in each flock?

Ans. The numbers were 18, 24, 126, 1770, respectively.

24. A body of men are just sufficient to form a hollow equilateral wedge, three deep; and if 597 be taken away, the remainder will form a hollow square, four deep, the front of which contains one man more than the square root of the number contained in a front of the wedge. What is the number of men?

Ans. 693.

25. Two men, *A* and *B*, undertook to perform a piece of work in four days, for which they were to receive a certain number of shillings; but after some time, finding that they should not be able to finish it in the time proposed, they called in *C* to assist them; and upon an equitable division of the money, *C* received a sum equal to the square root of the whole number of shillings; but had they been obliged to call in *C* to their assistance $1\frac{1}{9}$ day sooner, his share of the money would have been two-fifths more. How long did *C* work, and what did he receive?

Ans. He worked 2 days, and received 5 shillings.

26. A cask, whose content is 20 gallons, is filled with brandy, a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water; after which the first cask is filled with the mixture, and it appears, that if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off.

Ans. 10 gallons.

27. There are three numbers, the difference of whose differences is 5; their sum is 20; and their continual product 130. Required the numbers.

Ans. 2, 5, and 13.

28. There are three numbers, the difference of whose differences is 3; their sum is 21; and the sum of the squares of the greatest and least is 137. Required the numbers.

Ans. 4, 6, 11.

29. There is a number consisting of 2 digits, which when divided by the sum of its digits gives a quotient greater by 2 than the first digit. But if the digits be inverted, and then divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Required the number.

Ans. 24.

30. *A* and *B* gained by trading £100. Half of *A*'s stock was less than *B*'s by £100; and *A's* gain was three-twentieths of *B*'s stock. What did each put into stock, and what are the respective shares of the gain?

Ans. *A*'s stock was £600, and *B*'s £400. *A*'s gain was £60, and *B*'s £40.

31. *A*, *B*, and *C* were three Architects. *A* and *B* built four warehouses with flat roofs, each a large one, and each a small one; the linear width of the two large ones being the same, and also that of the two small ones. *A* built his as long and as high as they were wide; but *B* made the length and height of his large one equal to the width of his small one, and the length and height of his small one equal to the width of his large one, in such a manner that the difference between the solid content of those built by *A*, and those built by *B*, was 73728 cubic feet. *C* also built a warehouse upon a square plot of ground, which was equal to the difference between the ground-plots occupied by those which *A* built, and found that it would have stood on 2688 square feet, if he had added eight times as many square feet to the ground-plot as there were linear feet in its width. How many feet wide were the several buildings erected by *A*, *B*, and *C*?

Ans. The width of *A*'s and *B*'s large warehouse was 52 feet, and of their small one 20: and the width of *C*'s 48.

32. A certain sum was to be raised on three estates belonging to A, B, and C, at the rate of one shilling *per* acre. Now the number of acres A and B had, were as 3 to 7; and if the number of acres in the whole were divided by one-third of the product of the numbers in the first and third, the quotient would be $\frac{3}{4}$. Also the sum paid by A and C was 36 shillings less than the sum of three times the money paid by C, and two-sevenths of the money paid by B. Of how many acres did each estate consist; and what was the whole sum to be raised?

Ans. A had 12, B 28, and C 20 acres; and the sum was £3.

33. A Butcher bought a certain number of calves and sheep, and for each of the former gave as many shillings as there were sheep, and for each of the latter one-fourth as much. Now had he given 4 shillings more for each of the former, and 2 shillings more for each of the latter, he would have paid seven pounds more. But had a sheep cost as much as a calf, he would have expended £56. 8*s*. How many did he buy of each; and what were their prices?

Ans. 23 calves, and 24 sheep; and their prices were 24, and 6 shillings, respectively.

34. A Farmer at a fair found the price of an ox equal to that of three sheep, and that he could just dispose of £100, buying twice as many sheep as oxen. But waiting till the evening, when the price of an ox fell £1, and of a sheep 6*s*. 8*d*., he got for £100 three times as many sheep as oxen, and increased his whole stock by ten more than he would have done in the former case. How many sheep and oxen did he buy, and what was the price of each?

Ans. 10 oxen and 30 sheep; and the prices were £5, and £1. 13*s*. 4*d*.

35. Two persons, A and B, comparing their wages, observe that if A had received *per* day in addition to what he does receive, a sum equal to one-fourth of what B received *per* week,

and had worked as many days as B received shillings *per* day, he would have received £2. 8*s.*; and had B received 2 shillings a day more than A did, and worked for a number of days equal to half the number of shillings he received *per* week, he would have received £4. 18*s.* What were their daily wages?

Ans. A's 5 shillings, and B's 4.

36. There are four towns in the order of the letters, A, B, C, D. The difference between the distances from A to B and from B to C is greater by four miles than the distance from B to D. Also the number of miles between B and D is equal to two-thirds of the number between A and C. And the number between A and B is to the number between C and D as seven times the number between B and C : 26. Required the respective distances.

Ans. $AB = 42$, $BC = 6$, $CD = 26$ miles.

37. A person bought a quantity of cloth of two sorts for £7. 18*s.* For every yard of the better sort he gave as many shillings as he had yards in all; and for every yard of the worse as many shillings as there were yards of the better sort more than of the worse. And the whole price of the better sort was to the whole price of the worse as 72 to 7. How many yards had he of each?

Ans. 9 yards of the better, and 7 of the worse.

38. From each of two bags containing a certain number of balls respectively, a person draws out a handful, and finds that the number remaining in the greater is exactly the cube of that remaining in the lesser, and exactly the square of one handful. He then draws out of the greater, until he finds that the number remaining in it is exactly the square of that remaining in the lesser, and also that if he now empties the greater into the lesser, its original number will be increased by two-thirds. Determine the number of balls in each bag.

Ans. 72 and 12.

39. A Farmer sold a certain number of bushels of barley, and ten bushels of wheat, for £7. 19*s.* Now each bushel of

wheat cost within 3 shillings as much as two bushels of barley. He afterwards sold as many bushels of barley and four more, and fifteen bushels of wheat, and received two shillings *per* bushel more for his wheat and barley than he did before; when he found that if he had received £1. 4*s*. more, he should just have received twice as much as he did before. How many bushels of barley did he sell the first time; and what were the prices *per* bushel of the wheat and barley?

Ans. 7 bushels of barley; and the prices of wheat and barley were 11*s*. and 7*s*. *per* bushel.

40. A Farmer laid up a stock of corn, expecting to sell it in six months at three shillings *per* bushel more than he gave for it. But the price of corn falling one shilling *per* bushel, he found that by selling it he should lose the price of five bushels. He therefore kept it till the end of the year, and selling it at two shillings *per* bushel under prime cost, found his loss to be ten shillings less than his expected gain. Required the quantity of corn laid up, and the price *per* bushel, allowing 5 *per cent.* simple interest.

Ans. 40 bushels, and the price was 10*s*. *per* bushel.

41. In digging among some ruins, the workmen found 9 urns, together containing 60 gold coins; the second and eighth containing 8 and 4 respectively. They secreted a certain number of these, greater than the number they left; which being afterwards recovered, it was found that the number of urns secreted was to the number left as the number of coins secreted was to the number remaining. Now if instead of taking the second urn they had carried off the eighth, then the number of coins taken away would have been to the number remaining as the square of the number of urns secreted to the difference between that square and 20 times the number of urns remaining. Required the numbers of urns and coins secreted.

Ans. 6 urns, and 40 coins.

42. Two men, *A* and *B*, set out from the same place to travel. *A* goes in 6 days twice as many miles as *B* goes in

5 days, but does not arrive at the end of his journey till 5 days after B has arrived at the end of his, when he finds that he has travelled 259 miles more than B. But had B gone 2 miles *per* day more than he did, and A stopped 6 days sooner, A would then have gone only 37 miles more than B. How many miles did each travel *per* day, and how many days did they travel?

Ans. A travelled 11 days, and 35 miles *per* day; B travelled 6 days, and 21 miles per day.

43. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask. After that Silenus awoke, and drank what Bacchus had left. Had they drunk both together, it would have been emptied two hours sooner, and Bacchus would have drunk only half what he left Silenus. Required the time in which they would empty the cask separately.

Ans. Silenus in 3 hours, and Bacchus in 6.

44. Two persons, A and B, comparing the distances they have travelled, found that the square of the number of miles which A usually walked *per* hour, exceeded the square of the number which B usually walked by 5; and that if to the square of the product of those numbers there be added the square of the sum of their fourth powers, augmented by the product of the square of the difference of their squares into the square of the product of the numbers themselves, the aggregate amount would be 10345. How many miles did each walk *per* hour?

Ans. A walked 3, and B 2 miles.

45. From the middle of a town two streets branched off, and crossed a river that ran in a straight course, by two bridges A and B. From their junction a sewer equally inclined to both streets led to a point in the river at the distance of 6 chains from the bridge A, and a distance from B less by 11 chains than the length of the sewer: the expense of making it amounting to as many pounds *per* chain, as there were chains in the street leading to A. The sewer however being insufficient to carry off

the water, an additional drain was made from a point in this street, distant 4 chains from the bridge *A*, which entered the river at the same point with the sewer, and was equally inclined to the river and sewer. Now it was found that a drain down the middle of each street, at the rate of £9 *per* chain, would have cost only £54 more than the expense of the sewer. Required the lengths of the streets and the sewer; and the distance of its mouth from the bridge *B*.

Ans. The lengths of the streets were 18 and 30 chains, of the sewer 21, and the distance from *B* 10.

46. Two plantations, one of an oblong, and the other of a square form, contain the same number of trees, and they have one fence common to both, viz. that which bounds the end of the oblong one. Upon every pole in the square are planted as many trees as there are poles in the square, and upon every pole in the oblong four times as many trees as there are poles in the breadth, besides 144 in the hedges. Also the area of the oblong wants 6 poles to be to the area of the square as 3 to 2. Required the number of trees.

Ans. 1296.

47. The roof of a storehouse is formed of two squares terminated by two equal and parallel isosceles triangles; the height of the walls being equal to the base of either of these triangles. The quantity of wood which the storehouse will hold, increased by six cubical piles each of the same length as the building, is to the quantity which the same storehouse would hold if its roof were flat, in the proportion of 11 : 2. The roof cost as many pence *per* square foot, as there are feet in its ridge, and the flooring was laid at the same rate. Both together cost £208. 6*s*. 8*d*. Required the dimensions of the storehouse.

Ans. The length is 25, and the height 15 feet.

48. There are two sorts of metal, each being a mixture of gold and silver, but in different proportions. Two coins from these metals of the same weight are to each other in value as

11 to 17; but if to the same quantities of silver as before in each mixture double the former quantities of gold had been added, the values of two coins from them of equal weights would have been to each other as 7 to 11. Determine the proportion of gold to silver in each mixture, the values of equal weights of gold and silver being as 13 to 1.

Ans. The proportion of gold to silver is 1 : 9 in the first mixture, and 1 : 4 in the second.

49. A Mason has two cubical pieces of white marble of exactly the same size, and two cubical equal pieces of black, larger than the other. The number of solid yards in the four pieces is 9 more than 11 times the number of yards in a side of a white one, together with 12 times the number in a side of a black one. He afterwards finds another block, the length of which is two yards longer than a side of one of the white pieces, and the width 4 times the length of a side of the black one; and this when laid on its largest side occupies a space greater by 3 yards than the difference between 4 times the space occupied by a black, and 3 times the space occupied by a white one. Required the dimensions of the blocks.

Ans. The side of a white block is 1 yard, and of a black one 3 yards; the length of the other is 3 yards, and the width 12 yards.

50. Two Labourers, A and B, whose rates of working were as 3 : 5, were employed to dig a ditch. A worked 12 hours, and B 10 hours a day. B being called away, A worked one day alone, in order to complete the work. When they were paid, B received as many pence more than A as the number of days they worked together. Now had B been called away a day sooner, A would have received 3*s.* 11*d.* more than B at the conclusion of the work. Required their respective daily wages, on supposition that the payment to each was in proportion to the work performed.

Ans. A's daily wages were 18*d.*, and B's 25*d.*

51. A Steam-boat sets out from London 3 miles behind a wherry; and having got to the same distance a-head, it over-

takes a barge floating down the stream, and reaches Gravesend $1\frac{1}{2}$ hours afterwards. Having waited in order to land the passengers $\frac{1}{4}$th of the time of coming down, it starts to return, and meets the wherry in three-quarters of an hour, the barge being then $5\frac{1}{4}$ miles a-head of the steam-boat, and arrives at London at the same time that the wherry was in coming down from thence. Required the distance between London and Gravesend, and the rate of each vessel.

Ans. 30 miles, and the rates 9 and 3 miles an hour.

52. *A* and *B* travelled on the same road and at the same rate from Huntingdon to London. At the 50th milestone from London, *A* overtook a drove of geese which were proceeding at the rate of three miles in 2 hours; and two hours afterwards met a stage waggon, which was moving at the rate of nine miles in four hours. *B* overtook the same drove of geese at the 45th milestone, and met the same stage waggon exactly forty minutes before he came to the 31st milestone. Where was *B* when *A* reached London?

Ans. 25 miles from London.

53. The hold of a vessel partly full of water (which is uniformly increased by a leak) is furnished with two pumps worked by *A* and *B*, of whom *A* takes three strokes to two of *B*'s; but four of *B*'s throw out as much water as five of *A*'s. Now *B* works for the time in which *A* alone would have emptied the hold; *A* then pumps out the remainder, and the hold is cleared in 13 hours and 20 minutes. Had they worked together, the hold would have been emptied in 3 hours and 45 minutes; and *A* would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak.

Ans. The quantity in the hold was 1200 gallons, and the horary influx 120 gallons.

X. *Problems in Arithmetical and Geometrical Progressions.*

1. THERE are three numbers in arithmetical progression, whose sum is 21; and the sum of the first and second is to the sum of the second and third as 3 to 4. Required the numbers.

Ans. 5, 7, 9.

2. There are six towns in the order of the letters, A, B, C, D, E, F, whose distances from each other are in an increasing arithmetical progression. The distance from A to C is 16 miles, and from C to E is 24 miles. Required their respective distances.

Ans. From A to B is 7, from B to C 9, from C to D 11, from D to E 13, and from E to F 15 miles.

3. A person makes a mixture of 51 gallons, consisting of brandy, rum, and water, the quantities of which are in arithmetical progression. The number of gallons of brandy and rum together is to the number of gallons of rum and water together as 8 to 9. Required the quantities of each.

Ans. 15 gallons of brandy, 17 of rum, and 19 of water.

4. A number consisting of three digits which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 48; and if 198 be subtracted from it, the digits will be inverted. Required the number.

Ans. 432.

5. During a scarcity, a person wished to make a mixture of 24 bushels, consisting of wheat, oats, and barley, the quantities of each forming an increasing arithmetical progression. Not being able however to procure any barley, he mixed additional quantities of wheat and oats in the proportion of 2 to 3, so as to complete his 24 bushels, when he found the whole quantities of wheat and oats to be in the proportion of 5 to 7. How many bushels of each did he originally intend to mix?

Ans. 6 of wheat, 8 of oats, and 10 of barley.

6. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers?

Ans. 54, 18, 6, and 2.

7. A person employed three workmen, whose daily wages were in arithmetical progression. The number of days they worked was equal to the number of shillings that the second received *per* day. The whole amount of their wages was seven guineas, and the best workman received 28 shillings more than the worst. What were their daily wages?

Ans. 5, 7, and 9 shillings.

8. There are three numbers in geometrical progression; the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

Ans. 3, 6, 12.

9. There are three numbers in geometrical progression; whose sum is 14; and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

Ans. 2, 4, 8.

10. There are three numbers in geometrical progression, whose continued product is 64, and the sum of their cubes is 584. Required the numbers.

Ans. 2, 4, 8.

11. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers.

Ans. 1, 3, 9, 27.

12. From two towns which were 168 miles distant, two persons, *A* and *B*, set out to meet each other; *A* went 3 miles the first day, 5 the next, 7 the third, and so on; *B* went 4 miles the first day, 6 the next, and so on. In how many days did they meet?

Ans. 8.

13. A traveller set out from a certain place, and went 1 mile the first day, 3 the second, 5 the next, and so on, going every day 2 miles more than he had gone the preceding day. After he had been gone three days, a second sets out, and travels 12 miles the first day, 13 the second, and so on. In how many days will the second overtake the first?

Ans. In 2, and 9 days.

14. A person has two pieces of ground, one of which is in the form of an equilateral triangle, and the other of a rectangular parallelogram, one side of which is equal to a side of the triangle, and the other side is 8 yards less. These he plants with trees at the distance of two yards from each other, and finds that there are 5 more on the rectangle than on the triangle. What are the lengths of the sides?

Ans. A side of the triangle is 20 yards, and the sides of the parallelogram are 20 and 12 yards.

15. There are four numbers in arithmetical progression, whose sum is 28; and their continual product is 585. Required the numbers.

Ans. 1, 5, 9, 13.

16. There are four numbers in arithmetical progression; the sum of the squares of the first and second is 34; and the sum of the squares of the third and fourth is 130. Required the numbers.

Ans. 3, 5, 7, 9.

17. The sum of £700 was divided among four persons, whose shares were in geometrical progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

Ans. £108, £144, £192, £256.

18. Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre;

and they altogether can finish the undertaking in 60 days. In how many days can each separately reap an acre?

Ans. 2, 3, 4, 5, 6 days.

19. Out of a vessel containing 24 gallons of pure spirit, a vintner drew off at three successive times a certain number of gallons, which formed an increasing arithmetical progression, in which the difference between the squares of the extremes was equal to 16 times the mean, and filled up the vessel with water after each draught, till he found what he last drew off reduced to one-sixth of its original strength. Required the number of gallons of pure spirit drawn off each time.

Ans. 12, 8, $3\frac{1}{3}$.

20. A number of persons purchased a field for £345. The youngest contributed a certain sum, the next £5 more, the third £5 more than the second, and so on to the oldest. For the greater accommodation of the seniors, the field was divided into two parts, the younger half taking a portion proportional to the sum they had subscribed; and in order that each might have an equal share in this portion, they agreed to equalize their contributions, and each to pay £22. Required the number of persons and the sums paid by each.

Ans. The number of persons was 10; and the sum paid by the youngest £12.

21. The number of deaths in a besieged garrison amounted to 6 daily; and allowing for this diminution their stock of provisions was sufficient to last for 8 days. But on the evening of the sixth day 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the stock of provisions unconsumed at the end of the sixth day to support 6 men for 61 days; it is required to find how long it would support the garrison; and the number of men alive when the provisions were exhausted.

Ans. 6 days, and 26 men remained alive when the provisions were exhausted.

22. A Ship with a crew of 175 men set sail with a store of water sufficient to last to the end of the voyage. But in 30 days the scurvy made its appearance, and carried off three men every day, and at the same time a storm arose, which protracted the voyage three weeks. They were however just enabled to arrive in port, without any diminution in each man's daily allowance of water. Required the time of the passage, and the number of men alive when the vessel reached harbour.

Ans. The voyage lasted 79 days, and the number of men alive was 28.

23. Three persons, *A*, *B*, and *C*, went into a gaming house; the sums which they severally had, were in a decreasing geometrical progression. Upon quitting it they found that the sums which they then had, were in a decreasing arithmetical progression; that what *B* had remaining was to what he had lost in proportion of the sum to the difference of what he and *C* had at first; and that *C* had neither won nor lost. If *C* had won what *A* lost, he would then have had £64 more than *A* had remaining; also the whole sum which they had remaining was to that they had lost as 6 : 7. Required the sums which they had at first.

Ans. 144, 48 and 16 pounds respectively.

24. The Fly starts 10 miles before the Telegraph; but the Fly coachman having made an appointment with the driver of the Telegraph, walks his horses so as to be overtaken at the end of the second mile. Now it is observed, that the number of revolutions made in a given time by the hinder wheel of the Fly, its fore wheel, and the hinder wheel of the Telegraph increase in arithmetical progression, and that the circumference of these wheels, viz. of the fore wheel of the Fly, its hinder wheel, and the hinder wheel of the Telegraph, increase in a geometrical progression, whose common ratio is the same as the common difference of the arithmetical progression. It is required to find the ratio that the wheels bear to each other.

Ans. 1, 2, 4 are their proportional lengths.

25. A company of Merchants fitted out a privateer, each subscribing £100. The captain subscribed nothing, but was entitled to a £100 share at the end of every certain number of months. In the course of 25 months he captured three prizes, which were in geometrical progression, the middle term being one-fourth of the cost of the equipment, the common ratio the number of months which entitled the captain to his £100 share, and their sum £1375 more than the cost of the equipment. After deducting £875 for prize-money to the crew, the captain's share of the remainder amounted to one-fifth of that of the company. Required the number of merchants, and the captain's pay.

> *Ans.* The number of merchants was 25, and the captain was entitled to a £100 share at the end of every 5 months.

26. On the institution of Savings Banks, an industrious labourer with his wife and children saved each a certain number of pence in a decreasing arithmetical progression. The sum saved monthly, was less by 3*s*. 3*d*. than would have purchased one-sixth of as many bushels of wheat as the seventh child saved pence: the price of wheat being such that the sum saved by the eldest and fifth child augmented by 10*s*. would buy two bushels. But wheat rising 2*s*. *per* bushel, and work being scarce, the family find the sum saved would not buy as much wheat as their former savings by two bushels; when it appears that at this rate the sum annually saved would be less by five guineas than by the former. Now the two youngest dying, it is found that if the remaining members of the family saved each one shilling less than the oldest child had done before the rise of wheat, their monthly account with the bank would not be affected by the deaths of the two youngest: but if they saved only 2*d*. less than the oldest had done, their monthly account would be 2*s*. 1*d*. less than it was at the first institution.

Of how many did the family consist? What were the sums saved by each? and what was the price of wheat?

Ans. The family at first consisted of 10. The labourer saved 4*s.*, each member saving 3*d.* less than the preceding. And the price of wheat was 8*s.* *per* bushel.

XI. *Equations and Problems where there are more unknown Quantities than independent Equations.*

1. Given $3x + 5y = 26$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 7, & y = 1, \\ x = 2, & y = 4. \end{cases}$$

2. Given $5x + 8y = 153$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 29, & y = 1, \\ x = 21, & y = 6, \end{cases} \begin{cases} x = 13, & y = 11, \\ x = 5, & y = 16. \end{cases}$$

3. Given $5x + 21y = 2000$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 379, & y = 5, \\ x = 358, & y = 10; \end{cases}$$

with 17 other corresponding pairs, the values of x decreasing by 21, and those of y increasing by 5.

4. Given $8x + 3y = 17$, find corresponding positive integral values of x and y.

Ans. $x = 1$, $y = 3$.

5. Given $10x + 17y = 71$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 2, \\ y = 3. \end{cases}$$

6. Given $11x + 7y = 108$, find corresponding positive integral values of x and y.

Ans. $x = 6$, $y = 6$.

7. Given $11x + 15y = 1031$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 91, \text{ and } y = 2, \\ x = 76, \text{ and } y = 13; \end{cases}$$

with five other corresponding pairs, the values of x decreasing by 15, and those of y increasing by 11.

8. Given $13x + 7y = 141$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 6, \\ y = 9. \end{cases}$$

9. Given $13x + 14y = 200$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 10, \\ y = 5. \end{cases}$$

10. Given $17x + 7y = 310$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 3, \\ y = 37, \end{cases} \begin{cases} x = 10, \\ y = 20, \end{cases} \begin{cases} x = 17, \\ y = 3. \end{cases}$$

11. Given $27x + 16y = 1600$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 16, \\ y = 73, \end{cases} \begin{cases} x = 32, \\ y = 46, \end{cases} \begin{cases} x = 48, \\ y = 19. \end{cases}$$

12. Given $71x + 17y = 1005$, find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 12, \\ y = 9. \end{cases}$$

13. Given $99x + 19y = 1900$; find corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 19, \\ y = 1. \end{cases}$$

14. Given $5x - 7y = 3$; find the least corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 2, \\ y = 1. \end{cases}$$

15. Given $7x - 12y = 19$; find the least corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 13, \\ y = 6. \end{cases}$$

16. Given $11x - 18y = 63$; find the least corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 9, \\ y = 2. \end{cases}$$

17. Given $13x - 17y = 54$; find the least corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 12, \\ y = 6. \end{cases}$$

18. Given $19x - 117y = 11$; find the least corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 56, \\ y = 9. \end{cases}$$

19. Given $14x - 5y = 7$; find the least corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 3, \\ y = 7. \end{cases}$$

20. Given $17x - 7y = 1$; find the least corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 5, \\ y = 12. \end{cases}$$

21. Given $24x = 13y + 16$; find the least corresponding positive integral values of x and y.

$$\text{Ans.} \begin{cases} x = 5, \\ y = 8. \end{cases}$$

22. Given $3x + 7y + 17z = 100$; find all the positive integral values of x, y, and z, which satisfy the equation.

$$\text{Ans.} \begin{cases} x = 23, & x = 16, \\ y = 2, & y = 5, \quad \text{\&c.} \\ z = 1. & z = 1, \end{cases}$$

23. Given $10x + 11y + 12z = 300$; find all the positive integral values of x, y, and z, which satisfy the equation.

$$\text{Ans.} \begin{cases} x = 20, & x = 21, & x = 22, \\ y = 8, & y = 6, & y = 4, \quad \text{\&c.} \\ z = 1. & z = 2. & z = 3, \end{cases}$$

24. Given $17x + 23y + 3z = 200$; find all the positive integral values of x, y, and z, which satisfy the equation.

$$\text{Ans.} \begin{cases} x = 3, & x = 2, & x = 1, & x = 6, \\ y = 1, & y = 2, & y = 3, & y = 1, \quad \text{\&c.} \\ z = 42. & z = 40. & z = 38. & z = 25, \end{cases}$$

25. Given $20x + 15y + 6z = 171$; find all the positive integral values of x, y, and z, which satisfy the equation.

$$\text{Ans.} \begin{cases} x = 3, & x = 6, \\ y = 1, & y = 1, \quad \text{\&c.} \\ z = 16. & z = 6, \end{cases}$$

26. Given $35x + 43y + 55y = 4000$; find all the positive integral values of x, y, and z, which satisfy the equation.

$$\text{Ans.} \begin{cases} x = 105, & x = 85, & x = 82, \\ y = 5, & y = 20, & y = 25, \quad \text{\&c.} \\ z = 2. & z = 3. & z = 1, \end{cases}$$

27. $\left.\begin{array}{r} 6x + 7y + 4z = 122, \\ 11x + 8y - 6z = 145, \end{array}\right\}$ to find the corresponding positive integral values of x, y, and z.

Ans. $x = 9$, $y = 8$, $z = 3$.

28. Given $\left.\begin{array}{r} 3x + 5y + 7z = 560, \\ 9x + 25y + 49z = 2920, \end{array}\right\}$ find the corresponding positive integral values of x, y, and z.

$$\text{Ans.} \begin{cases} x = 15, & y = 82, & z = 15, \\ x = 50, & y = 40, & z = 30. \end{cases}$$

29. Required the positive integral solutions of the equation $2xy + x + y = 195$.

$$\text{Ans.} \begin{cases} x = 8, & x = 11, \\ y = 11. & y = 8. \end{cases}$$

30. Required the positive integral solutions of the equation $3xy - 4y + 3x = 14$.

$$\text{Ans.} \begin{cases} x = 3 \text{ and } 2, \\ y = 1 \text{ and } 4. \end{cases}$$

31. Required the positive integral solutions of the equation $5xy = 2x + 3y + 18$.

$$\text{Ans.} \begin{cases} x = 5, \\ y = 10. \end{cases} \begin{cases} x = 3, \\ y = 2. \end{cases} \begin{cases} x = 7, \\ y = 1. \end{cases}$$

32. Required the positive integral solutions of the equation $7xy - 5x = 3y + 39$.

$$\text{Ans.} \begin{cases} x = 1,\ 3,\ 5,\ 21. \\ y = 11,\ 3,\ 2,\ 1. \end{cases}$$

33. Required the integral solutions of the equation $2xy - 3x^2 + y = 1$.

$$\text{Ans.} \begin{cases} x = 3, \\ y = 4. \end{cases}$$

34. Find the least whole number which divided by 3 and 7 leaves remainders 1 and 2.

Ans. 16.

35. Find a number which divided by 6 leaves a remainder 2, and divided by 13 leaves a remainder 3.

Ans. The least number is 68.

36. Find the least whole number which divided by 17 shall leave a remainder 7, but being divided by 26 the remainder shall be 13.

Ans. 143.

37. Find the least whole number which being divided by 28 will leave a remainder 17, and being divided by 19 will leave a remainder 13.

ANS. 241.

38. What number is that which divided by 23 gives a remainder 22, and being divided by 37 gives a remainder 36?

ANS. 850.

39. Find a number which divided by 39 leaves a remainder 16, and divided by 56 leaves a remainder 27.

ANS. 1147.

40. Find two fractions whose denominators shall be 7 and 9, and their sum $\frac{19}{21}$.

ANS. $\frac{4}{7}$, and $\frac{3}{9}$.

41. Find a number which divided by 2, 3, 5, respectively, leaves as remainders 1, 2, 3.

ANS. 23, 53, 113, &c.

42. Find a number which divided by 4, 5, 6, respectively, shall leave 3, 3, and 5 for remainders.

ANS. 23, 83, 143, &c.

43. Find a number such that when divided by 11 there shall be a remainder 3; when divided by 19 there shall be a remainder 5; and when divided by 29, a remainder 10.

ANS. The least number is 4128.

44. Find the least number which being divided by 28, 19, and 15, leaves remainders 13, 2, and 7.

ANS. 97.

45. Find the least number which divided by 6, 5, 4, 3, and 2, respectively, shall leave 5, 4, 3, 2, and 1, respectively, remaining.

ANS. 59.

46. Find the least number which divided by 3, 5, 7, and 2, shall leave remainders 2, 4, 6, and 0, respectively.

Ans. 104.

47. Find the least whole number which divided by 16, 17, 18, 19, and 20, shall leave 6, 7, 8, 9, and 10 remainders respectively.

Ans. 232550.

48. Find the least whole number which being divided by the nine digits respectively, shall leave no remainder.

Ans. 2520.

49. Divide 100 into two such parts that the one may be divisible by 7 and the other by 11.

Ans. 56 and 44.

50. Divide 100 into two such parts, that dividing the first by 5 there may remain 2, and dividing the second by 7 the remainder may be 4.

Ans. $\begin{cases} 82 \text{ and } 18, \\ 47 \text{ and } 53, \\ 12 \text{ and } 88. \end{cases}$

51. In how many different ways may £1000 be paid in crowns and guineas?

Ans. 190 different ways.

52. In how many different ways can £43. 10*s*. 6*d*. be paid with half-guineas and sovereigns?

Ans. In 3 different ways;

paying 81 half-guineas and 1 sovereign
" 41 " " 22 "
" 1 " " 43 "

53. In how many different ways is it possible to pay a bill of £351 with guineas and pieces of 27*s*. each?

Ans. In 36 different ways;

paying 333 guineas and one piece of 27*s*., and diminishing the former number by 9, and increasing the latter by 7.

54. In how many different ways is it possible to pay £20 in half-guineas and half-crowns?

Ans. 7 different ways.

55. In how many ways can £100 be paid in guineas and pistoles (17*s.* each)?

Ans. There are 6 different ways of payment;
paying 7 guineas and 109 pistoles,
" 24 " 88 "
increasing each time the number of guineas by 17, and diminishing the number of pistoles by 21.

56. A Gentleman has to pay £1000, and has only two sorts of coins, guineas valued at 21*s.* 6*d.* and Louis d'ors at 17*s.* each. How many different ways may the payment be made with these coins?

Ans. 27 different ways;
paying 32 of the former and 1136 of the latter, and increasing the former by 34 and diminishing the latter by 43, in each case.

57. If I have 9 half-guineas and 6 half-crowns in my purse, how may I pay a debt of £4. 11*s.* 6*d.*?

Ans. By 8 half-guineas and 3 half-crowns.

58. In how many ways can an equivalent for 13 dollars, at 3*s.* each, be given in English crowns and seven-shilling pieces?

Ans. Only *one*, viz. 5 crowns and 2 seven-shilling pieces.

59. A company of men and women spend £50. The men pay each 19*s.*, and each woman 13*s.* How many men and women are there?

Ans. 2 and 74; or 15 and 55;
or 28 and 36; or 41 and 17.

60. A Farmer laid out the sum of £1770 in purchasing horses and oxen. He paid £31 for each horse, and £21 for each ox. How many horses and oxen did he buy?

Ans. 9 horses and 71 oxen,
or 30 " and 40 "
or 51 " and 9 "

61. Two women have together 100 eggs. One says to the other, When I count mine by eights, there is an overplus of 7. The second replies, If I count mine by tens, I find the same overplus of 7. How many eggs had each?

ANS. The first had 63, and the second 37;
or the first had 23, and the second 77.

62. What quantities of tobacco at 16*d.* and 10*d.* *per* pound, may be mixed with 50 pounds of tobacco at 8*d.* *per* pound, so that the whole may be worth a shilling *per* pound?

ANS. 51 lbs. of the former, and 2 lbs. of the latter;
52 lbs. „ 4 lbs. „
and so on, increasing the former by 1, and the latter by 2.

63. *A* has moidores only; *B* only crowns. How can *B* pay *A* £1. 12*s.* most easily?

ANS. By paying 28 crowns, and receiving 4 moidores.

64. *A* having only one-pound notes, owes 12*s.* to *B*, who has only seven-shilling pieces. What is the least number which *A* and *B* must respectively interchange, so that the debt may be discharged?

ANS. He must give 2 notes, and receive 4 seven-shilling pieces.

65. A person buys some horses and oxen. He pays £31 for a horse, and £20 for each ox; and he finds that the oxen cost him £7 more than the horses. How many horses and oxen did he buy?

ANS. The least numbers are 3 and 5: and other answers may be obtained by adding 20 to the former, and 31 to the latter continually.

66. Find two numbers, such that their product added to their sum may be 79.

ANS. 1 and 39; 3 and 19; 4 and 15; and 9 and 7.

67. Find three integers, such that if the first be multiplied by 3, the second by 5, and the third by 7, the sum of the products may be 560. But if the first be multiplied by 9, the second by 25, and the third by 49, the sum of the products may be 2920.

ANS. 15, 82, and 15;
or 50, 40, and 30.

68. It is required to buy 20 fowls for 20 shillings; viz. geese at 4*s*., quails at 6*d*., and pigeons at 3*d*. each.

ANS. 3 geese, 15 quails, and 2 pigeons.

69. Thirty persons, men, women, and children, spend 50*s*. in a tavern; the share of each man is 3*s*., that of a woman 2*s*., and that of a child 1*s*. How many persons were there of each class?

ANS. 1 man, 18 women, and 11 children; and there are 8 other answers, the number of men and children increasing by 1, and the women decreasing by 2.

70. A Grazier bought calves, sheep, and pigs to the number of 100 for £100. The calves cost £3. 10*s*. each, the sheep £1. 6*s*. 8*d*., and the pigs 10*s*. How many had he of each?

ANS. 5	calves,	42	sheep, and	53	pigs;
or 10	,,	24	,,	66	,,
or 15	,,	6	,,	79	,,

71. Find three fractions with denominators 3, 4, and 5, of which the sum is $\frac{133}{60}$.

ANS. $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$.

72. A Vintner has wine at 2*s*., 1*s*. 10*d*., and 1*s*. 6*d*. *per* gallon; how much of each sort must he take, so as to make a mixture of 30 gallons to be sold at 1*s*. 8*d*. a gallon?

ANS. 2	of the first,	12	of the second, and	16	of the third;
4	,,	9	,,	17	,,
6	,,	6	,,	18	,,
8	,,	3	,,	19	,,

73. A person buys 100 head of cattle of *four* different kinds for £100. For the first sort he gives £10 each; for the second £5; for the third £2; and for the fourth 10*s*. How many of each did he buy?

ANS. 1	of the 1st,	1	of the 2nd,	24	of the 3rd, and	74	of the 4th;
1	,,	2	,,	21	,,	76	,,

of which there will be six more answers, the second increasing by 1, the fourth by 2, and the third decreasing by 3.

Also 4	of the 1st,	1	of the 2nd,	5	of the 3rd, and	90	of the 4th;
4	,,	2	,,	2	,,	92	,,

74. Find two square numbers, whose difference may be a square.

ANS. 1 and $\frac{16}{25}$; 1 and $\frac{9}{25}$; 1 and $\frac{64}{289}$, &c.;

or in integers, the squares of 5 and 4; 10 and 6; 17 and 8; 26 and 10, &c.

75. Find two square numbers, whose difference shall be equal to a given square (a^2).

ANS. $\frac{25a^2}{9}$ and $\frac{16a^2}{9}$; $\frac{25a^2}{16}$ and $\frac{36a^2}{64}$; $\frac{289a^2}{225}$ and $\frac{65a^2}{225}$, &c.;

or in integers, 25 and 16, if $a = 3$;
100 and 36, if $a = 8$;
289 and 64, if $a = 17$, &c.

76. Find a number, such that if unity be subtracted from twice its square, the remainder may be a square.

ANS. 1, 5, &c.

77. Find a value of x, which will make $6x^2 + 13x + 6$ a square.

ANS. 1 and $\frac{6}{5}$.

78. Determine n affirmative integers, such that the square of the greatest may be equal to the sum of the squares of all the rest.

ANS. If $n = 3$, the numbers will be 3, 4, 5;
$n = 4$, „ „ 2, 3, 6, 7, &c.

79. Determine two numbers which are not squares, but such that their product shall be a square; and also that product added to the square of either number shall be a square.

ANS. If x = any number, the required numbers will be

x and $\frac{9}{16}x$; x and $\frac{16}{9}x$; x and $\frac{225}{64}x$; x and $\frac{225}{144}x$, &c.

80. Determine two numbers, such that if either of them be subtracted from the square of the other, the remainder may be a square.

ANS. $\frac{9}{7}$ and $\frac{8}{7}$; $\frac{19}{11}$ and $\frac{15}{11}$; $\frac{12}{7}$ and $\frac{9}{7}$, &c.

81. Determine how many terms of the series of squares,

whose roots are 1, 2, 3, 4, 5, &c., must be taken, so that their sum may be a square number.

ANS. 24.

82. Divide a given square number (n^2) into two such parts, that the sum of their squares and the sum of their cubes may both be rational squares.

ANS. $\frac{8n^2}{23}$ and $\frac{15n^2}{23}$.

83. Determine two rational fractions, either of which being added to the square root of the other, shall give the same sum: and their difference shall be a perfect fourth power.

ANS. $\frac{1681}{6561}$ and $\frac{1600}{6561}$.

84. Determine two numbers, such that their sum, and the sum of the square of one and the cube of the other, may both be square numbers.

ANS. 28 and 8; 1176 and 49.

85. Determine two fractions, such that their sum and the sum of their squares may both be rational squares; and either of them added to the square of the other shall make the same square.

ANS. $\frac{2415}{4183}$ and $\frac{1768}{4183}$.

86. Determine three rational squares, whose sum shall be equal to their continual product.

ANS. $\frac{169}{64}$, $\frac{25}{36}$, and 4; $\left(\frac{2593}{1634}\right)^2$, $\left(\frac{65}{48}\right)^2$, and $\left(\frac{34}{31}\right)^2$.

87. Determine three numbers, such that their sum shall be a square, and the sum of their squares a perfect fourth power.

ANS. $(119)^2$, $2 \cdot (119) \cdot (180)$, and $2 \cdot (180)^2$, whose sum is $(349)^2$, and the sum of their squares $(281)^4$.

88. Determine three square numbers which shall be in arithmetical progression, but the sum of their square roots shall be a cube.

ANS. $(169)^2$, $(845)^2$, and $(1183)^2$.

APPENDIX I.

I. *Problems in Arithmetical Progression.*

1. DETERMINE the 28th term of the series 13, $12\frac{2}{3}$, $12\frac{1}{3}$, &c.

2. Having given the first and last terms of an arithmetic progression, and their common difference; determine their number of terms.

3. Having given the first and last terms of an arithmetic progression, and the number of terms; determine the progression, in general; and in the particular case where the first term is 100, the last 1, and the number of terms 19.

4. Find the series in arithmetic progression having 29 terms, of which the first is 3, and the last 17.

5. Having given the first term of an arithmetic series $= 2$, the number of terms $= 7$, and the common difference $= \frac{1}{3}$; find the last term.

6. Show that $\frac{4}{3}$, $\frac{1}{2}$, and $-\frac{1}{3}$, are in arithmetic progression; and find the 11th term of the series.

7. Show that $\frac{1}{12}$, $\frac{1}{8}$, $\frac{1}{6}$, are quantities in arithmetic progression; and find the sum of 8 terms of the series of which they are the first three terms.

8. In an arithmetic progression, it is observed that the fifth and ninth terms are 13 and 25: what is the 7th term?

9. The sum of the two first terms of an arithmetic progression is 4, and the 5th term is 9; find the series.

10. If three quantities are in an increasing arithmetic pro-

gression; show that the second will have to the first a greater ratio than the third to the second.

11. Find the sums of the following series:

$1 + 3 + 5 + 7 +$ &c. to n terms.

$1 + 5 + 9 + 13 +$ &c. to n terms.

$1 + 4 + 7 + 10 +$ &c. to 12 terms.

$5 + 7 + 9 + 11 +$ &c. to 50 terms.

$2 + 2\frac{1}{3} + 2\frac{2}{3} + 3 +$ &c. to 13 terms.

$\frac{1}{4} + \frac{3}{8} + \frac{1}{2} +$ &c. to 16 terms.

$\frac{1}{3} + \frac{7}{6} + 2 + \frac{17}{6} + \frac{11}{3} +$ &c. to 12 terms.

$\frac{5}{7} + 1 + 1\frac{2}{7} +$ &c. to 8 terms.

$-9 - 7 - 5 -$ &c. to 20 terms.

$-5 - 3 - 1$, &c. to 8 terms.

$11 + 8 + 5 +$ &c. to 8 terms.

$\frac{1}{2} - 1 - \frac{5}{2} -$ &c. to 29 terms.

$15 + \frac{44}{3} + \frac{43}{3} +$ &c. to 16 terms.

$\frac{5}{6} + \frac{1}{2} + \frac{1}{6} +$ &c. to 19 terms.

$\frac{27}{16} + \frac{18}{15} + \frac{57}{80} +$ &c. to 8 terms.

$\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} +$ &c. to n terms.

$na - b + (n-1).a + (n-2).a + b +$ &c. to n terms.

$(a+x)^2 + (a^2 + x^2) + (a-x)^2 +$ &c. to n terms.

$\left(\frac{1}{a} - \frac{n}{x}\right) + \left(\frac{1}{a} - \frac{n-1}{x}\right) + \left(\frac{1}{a} - \frac{n-2}{x}\right) +$ &c. to n terms.

12. Show that $\frac{a-b}{a+b} + \frac{3a-2b}{a+b} +$ &c. to n terms, is $= \frac{n}{a+b} \cdot \left(na - \frac{n+1}{2} \cdot b\right)$.

13. Show that the sum of the $(m - n)^{th}$ and $(m + n)^{th}$ terms of any arithmetical progression will be equal to twice the m^{th} term.

14. It is required to divide (a) into n parts proportional to the numbers 1, 2, 3, &c.

15. Having given the first and last terms, and the sum of an arithmetic series; determine the common difference, and apply it to the case where the first term is $= 1$, the last $= 50$, and the sum $= 204$.

16. Having given the first term $= 1$, the number of terms $= n$, and the sum $= s$, of an arithmetic series; determine the common difference.

17. Having given the n^{th} term of an arithmetic series, and also the sum of n terms; determine the series.

18. If the first term of an arithmetic series be $= 1$, and the common difference $= m$, the sum of n terms of the series is $\frac{1}{2} . \{mn^2 - (m - 2) . n$.

19. How many terms of the series $- 7 - 5 - 3 -$, &c. amount to 9200?

20. If the first term of an arithmetic series be $= 1$, the common difference $= 4$, and the sum $= 120$; determine the number of terms.

21. If the first term of an arithmetic series be $= 3\frac{1}{3}$, the common difference $= 1\frac{2}{3}$, and the sum $= 22$; determine the number of terms.

22. The first term of an arithmetic series is 3, the common difference is 4, and the sum of n terms is 1081; find n, and explain the double answer.

23. If the first term of an arithmetic series be $= 11$, the common difference $= - 5$, and the sum $= 5$; determine the number of terms.

24. There are five numbers, of which the first two are $2\frac{1}{3}$,

$3\frac{5}{24}$, and each number exceeds the preceding one by the same fraction; find the numbers and the sum of them.

25. The sum of an arithmetic series is $= 1455$, the first term $= 5$, and the number of terms $= 30$; determine the common difference.

26. The sum of an arithmetic series is 91, the common difference 2, and the last term 19; find the number of terms.

27. The sum of a certain number of terms of the series $21 + 19 + 17 +$ &c. is 120; find the last term and the number of terms.

28. If a be the first term, b the second, l the last term of an arithmetic series, the sum $= \frac{a+l}{b-a} \cdot \frac{l - 2a + b}{2}$.

29. In the expression $s = \{2a + (n-1) \cdot d\} \cdot \frac{1}{2}n$; if n be *negative*, point out the form of the series which satisfies *that* condition; and in finding n, show when the values, one, both, or neither are congruent values supplying interpretable solutions. Determine the series in the case of $a = 7$, $d = 2$, $s = 40$.

30. If a be the first and l the last term of an arithmetic series, in which the common difference is b, and the number of terms n; then will the sum of the series $= \frac{n \cdot (l^2 - a^2)}{2 \cdot (n-1) b}$.

31. The sum of n terms of any increasing arithmetic series, whose common difference is equal to the least term, will be equal to the sum of $(n + 1)$ magnitudes, each of which is half the greatest term of the progression.

32. If the number of terms of an arithmetic series be odd; show that the sum of the series is equal to the middle term multiplied by the number of terms.

33. The n^{th} term of an arithmetic progression is $\frac{3n-1}{6}$, and the sum of n terms is $\frac{n}{12} \cdot (3n + 1)$; find the series.

34. The first term is $n^2 - n + 1$, and the common difference is 2; prove that the sum of n terms is n^3; thence show that $1^3 = 1$, $2^3 = 3 + 5$, $3^3 = 7 + 9 + 11$, &c. If the first term be $n^{m-1} - n + 1$, the sum of n terms is n^m; and thence show that $1^4 = 1$, $2^4 = 7 + 9$, $3^4 = 25 + 27 + 29 +$ &c.

35. The sum of n terms of an arithmetic series is $pn + qn^2$, whatever be the value of n; find the m^{th} term.

36. In an arithmetic series the first term and the common difference are the same, and the sum of the series is always equal to the number of the terms + the square of that number; determine the series.

37. If from any square number (n^2) there be subtracted the sum of an arithmetic progression beginning from unity, having a common difference unity, and continued to as many terms as there are units in the root of the number (n); the remainder will be the sum of the progression continued to $(n - 1)$ terms.

38. The sum of an even number of terms of any arithmetic series whose common difference is equal to the least term, will be four times the sum of half that number of terms diminished by half the last term; the first term being the same in each case.

39. The latter half of $2n$ terms of an arithmetical series is equal to $\frac{1}{3}$rd of the sum of $3n$ terms of the same series.

40. Prove that 1, 3, 5, 7, &c. is the only arithmetic progression beginning from 1, in which the sum of the first half of any even number of terms bears to the sum of the second half the same constant ratio; and determine that ratio.

41. The sum of n terms of the series 1, 3, 5, 7, &c. is to the sum of $(n - 1)$ terms of the series 2, 4, 6, &c. $:: n : n - 1$; required a proof.

42. The difference between the sums of m and n terms of an arithmetic progression : the sum of $(m + n)$ terms $:: m - n : m + n$.

43. The two first terms of an arithmetic progression being together $= 18$, and the three next $= 12$; how many terms, beginning with the first, must be taken to make 28? and explain the reason of the double solution.

44. The first two terms of an arithmetic series being together $= 18$, and the third term $= 12$; how many must be taken to make 78?

45. The first and third terms of an arithmetic series being $2\frac{1}{4}$ and $1\frac{1}{2}$ respectively; find the second term and the sum of 19 terms.

46. The $(n+1)^{\text{th}}$ term of an arithmetic progression is $\dfrac{ma-nb}{a-b}$; required the sum of the series to $(2n+1)$ terms.

47. How many terms of the series 1, 3, 5, 7, &c. must be added together to produce the $(2m)^{\text{th}}$ power of a given quantity r?

48. Find n terms of the indefinite series 3, 5, 7, &c. whose sum may be the $(m)^{\text{th}}$ power of n.

49. In the two series 2, 5, 8, &c. and 3, 7, 11, &c., each continued to 100 terms; find how many terms are identical.

50. Having given (a) and (b), the $(m)^{\text{th}}$ and $(n)^{\text{th}}$ terms of an arithmetic series; determine the value of the $(x)^{\text{th}}$ term.

51. Having given as before; determine the sum of (p) terms of the series.

52. Having given (a) and (b), the $(m)^{\text{th}}$ and $(n)^{\text{th}}$ terms of an arithmetic series, and its last term $(a+b)$; determine the first term, common difference, and number of terms.

53. The sum of m terms of an arithmetical series is n, and the sum of n terms is m; show that the sum of $(m+n)$ terms is $-(m+n)$, and the sum of $(m-n)$ terms is $(m-n)\,.\left(1+\dfrac{2n}{m}\right)$.

54. If the m^{th} term be n, and the n^{th} term m; of how many

terms will the sum be $\frac{1}{2}(m+n).(m+n-1)$; and what will be the last of them.

55. In an arithmetic series, if the $(m+n)^{\text{th}}$ term $=p$, and the $(m-n)^{\text{th}}$ term $=q$; show that the $(m)^{\text{th}}$ term $=\frac{1}{2}.(p+q)$, and the $(n)^{\text{th}}$ term $=p-(p-q).\frac{m}{2n}$.

56. If x, y, and z be respectively the p^{th}, q^{th}, and r^{th} terms of an arithmetic progression; show that $(p-q).z+(r-p).y+(q-r).x=0$.

57. There are two series in arithmetic progression, the sums of which to n terms are $:: 13-7n : 3n+1$; prove that their first terms are as $3 : 2$, and their second terms as $-4 : 5$.

58. If S, S', S'' be the sums of three arithmetic series, $1=$ the first term of each, and the respective differences be $1, 2, 3$; prove that $S+S''=2S'$.

59. There is a series of n quantities in arithmetic progression whose first term is a, and common difference d; show that if there be taken n terms of the first, $n-1$ of the second, &c. to the n^{th} inclusive, the sum will be

$$\frac{n.(n+1)}{1.2.3}.\{3a+(n-1).d\}.$$

60. If S_1, S_2, S_3.... are the sums of n terms of different arithmetic series having the same first term, and common differences $1, 2, 3$.... respectively; show that S_1, S_2, &c. are in arithmetic progression; and when that first term is 1, prove that

$$S_1+S_2+\ldots\ldots+S_n=\frac{n^2.(n^2+3)}{4}.$$

61. If there be (p) arithmetical progressions, each beginning from unity, whose common differences are $1, 2, 3.....p$; show that the sum of their $(n)^{\text{th}}$ terms is

$$=\frac{1}{2}.\{(n-1).p^2+(n+1).p\}.$$

62. If a and d are respectively the first term and common difference of an arithmetic series, S_n the sum of n terms, S_{n+1}

the sum of $(n+1)$ terms, &c. prove that $S_n + S_{n+1} + S_{n+2}$ + &c. to n terms $= (3n-1) \cdot n \cdot \frac{a}{1 \cdot 2} + (7n-2) \cdot (n-1) \cdot n \cdot \frac{d}{1 \cdot 2 \cdot 3}$.

63. If S_1, S_2, S_3......S_p be the sums of (p) arithmetic progressions continued to n terms, and their first terms be 1, 2, 3, 4, &c. and their common differences 1, 3, 5, 7, &c.; show that $S_1 + S_2 + S_3 + \ldots\ldots + S_p = \frac{1}{2} \cdot (np+1)\, np$.

64. If S_1, S_2, S_3.....S_p are (p) arithmetic series each continued to n terms, whose first terms are the first (p) even numbers, and common differences the first (p) odd numbers respectively; show that

$$S_1 + S_2 + S_3 \ldots\ldots + S_p = np + \frac{n \cdot (n+1)}{2} p^2.$$

65. If S_1, S_2, S_3.....S_{2n} be the sums of n terms of $2n$ arithmetical progressions, whose first terms are the same, and common differences d, $2d$, $3d$....$2nd$; then will

$$(S_2 + S_4 \ldots\ldots + S_{2n}) - (S_1 + S_3 + \ldots\ldots + S_{2n-1}) = \tfrac{1}{2} n^2 \cdot (n-1) \cdot d.$$

66. If S_m denote generally the sum of m terms of any arithmetic progression; prove that

$$S_{n^2} = S_{2n} \cdot n \cdot \frac{n-1}{2} - S_n \cdot n \cdot (n-2).$$

67. Find three arithmetic means between 1 and 11; and seven between 1 and $-\frac{1}{3}$; and fifteen between 3 and 47.

68. If between all the terms of an arithmetic progression the same number of arithmetic means be inserted; show that the new series will still form an arithmetic progression.

69. Insert 6 arithmetic means between $\frac{1}{2}$ and $\frac{3}{4}$; and find their sum.

70. Insert n arithmetic means between a and b, and apply the general expression to insert 3 terms between 5 and $-\frac{1}{3}$; find the sum of the series for 20 terms.

71. There are n arithmetic means between 3 and 17, and the last is 3 times as great as the first; find the number of means.

72. There are n arithmetic means between 1 and 31, and the 7th mean is to the $(n-1)^{\text{th}}$ :: 5 : 9; prove that the number is 14.

73. Between a and b, a being less than b, insert n means such that $a_1 - a$, $a_2 - a_1$,......$b - a_n$ form an arithmetic progression whose common difference is d; and find the limits between which the value of d must lie.

74. The sum of n arithmetic means between 1 and 19 is to the sum of the first $(n-2)$ of them :: 5 : 3; determine the means.

75. If S be the sum of an arithmetic progression, a, b, c, d, &c. to n terms; determine the sum of the series

$$S \pm a,\ S \pm (a+b),\ S \pm (a+b+c), \text{ \&c.}$$

76. In any arithmetic progression of which a is the first term and $2a$ the common difference; prove that the number of terms which must be taken to make a sum S, is $\sqrt{\frac{S}{a}}$; S being assumed such that $\frac{S}{a}$ is any square number, but no other.

77. Determine the sum of n terms of the triangular numbers 1, 3, 6, 10, 15, &c. the terms of which series are 1, $1+2$, $1+2+3$, &c. the successive sums of 1, 2, 3, &c.

78. Determine the sum of n terms of the pyramidal numbers, 1, 4, 10, 20, 35, &c. the successive sums of 1, 3, 6, 10, 15, &c.

79. Find the sum of n terms of the series 1^2, 2^2, 3^2, &c.

80. Solve the equation $(x-1) + 2\,.\,(x-2) + 3\,(x-3)$ to 6 terms $= 14$.

81. Having given the first term and common difference

of an arithmetic progression; find the sum of n terms and the sum of their squares.

82. Having given the sum of $(2n)$ quantities in arithmetic progression, and the sum of their squares; determine the quantities themselves.

83. In the series 1, 2, 3, 4.....100, determine the sum of the numbers which are not squares.

84. Prove that the sum of the series $1^2 + 3^2 + 5^2$, &c. to n terms $= \frac{n}{3} \cdot (4n^2 - 1)$. And determine the sum of the series $a^2 + (a + b)^2 + (a + 2b)^2 +$ &c. to n terms.

85. The sum of the series 0, 1, 2, 3, &c. continued to an unknown number of terms, being $= 1225$; determine the sum of their squares.

86. The sum of 9 terms of the series $n^2 + (n + 1)^2 + (n + 2)^2 +$ &c. $= 501$; determine the value of n.

87. Determine the sum of 10 square numbers, whose roots are in an arithmetic progression, the least term of which is $= 3$, and the common difference $= 2$.

88. The square of any number of digits less than ten, each of which is unity, will, when reckoned from either end, form the same arithmetic series whose common difference is unity, and greatest term the number of digits in the root. Required a proof.

89. A man is employed in a certain manufacture, where the quantity of work which he produces in the 1st, 2nd, 3rd,.... days are 1, 3, 5, 7,.... For the first day's work he receives a shilling, and afterwards in proportion to his work. If a new workman is added every day under the same conditions, how much money will have been paid to the men after n days?

90. A gentleman owed to each of two persons, A and B, an equal sum of money, which he discharged as follows: to A he paid £8 the first payment, £12 the second, £16 the third,

and so continued increasing £4 each payment. Now B at his first payment received but £1, the second £4, the third £9, increasing according to the square of the number of payments. Determine what he owed each person, and the number of payments required to discharge the debt.

91. Compare the sum of the numbers 1, 2, 3, 4, &c. with the sum of their cubes.

92. Find the sum of n cube numbers, whose roots are in arithmetic progression, the least term of which is a, and common difference d.

93. Determine the arithmetic progression, the number of whose terms is 11, their sum 220, and the sum of their cubes 147400.

94. Prove that when n is indefinitely great,

$$\frac{a^r + (a+b)^r + (a+2b)^r + \text{\&c. to } n \text{ terms}}{n^{r+1}} = \frac{b^r}{r+1}.$$

95. Find the sum of (n) terms of a series of polygonal numbers, which numbers are formed by assuming any arithmetic series that has its first term 1, and difference a whole number, and by making generally the $(m)^{\text{th}}$ polygonal number equal to the sum of (n) terms of the arithmetical series.

96. If the first term of an arithmetic series be (a), the last term (l), the common difference (d); and $S_1, S_2, S_3 \ldots\ldots S_{m-1}$ be the sums of the 1st, 2nd, 3rd, ... $(m-1)^{\text{th}}$ powers of the terms; prove that $(l+d)^m - a^m = mdS_{m-1} + m \cdot \frac{m-1}{2}$

$$d^2 S_{m-2} + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot d^3 S_{m-3} + \text{\&c.}$$

97. Let a, $a-r$, $a-2r$, &c. and b, $b+r$, $b+2r$, &c. be two arithmetical progressions, each to n terms: then if the sum of the first $= 0$, and the sum of the product of the terms

of the first multiplied respectively by the corresponding terms of the second $= \frac{nr^2 . (1 - a^2)}{12}$; determine the value of n.

98. If the quantities a, b, c, d, &c. be in arithmetic progression; prove that the terms of any order of the differences of the quantities $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, &c. increase or decrease according as the progression decreases or increases.

99. A number (n) of boys arrange themselves in a right line at equal intervals, the nearest being at a given distance (a) from a fixed station S. A person P walks from S to the first boy; and as soon as he begins to return, the remaining boys move *from* S at the same rate as P, and stop when he comes to S. P advances again to the second boy; and whilst he is returning, the remainder move onward, and halt as before. The same is repeated until P has reached the last boy and returned. Now if under the same circumstances the boys had moved each time towards S, P would have passed over only $\frac{1^{th}}{m}$ part of his former distance. Show that the distance (d) of the boys from each other is equal to

$$\frac{(2^n - m - 1) . a}{n . (m + 1) - (2^n + m - 1)}.$$

100. The interior angles of a decagon are in arithmetic progression, and the least angle is $\frac{2}{3}$ of the greatest. Required the magnitude of every angle.

101. The interior angles of a rectilinear figure are in arithmetical progression. The least angle is 120°, and the common difference 5°. Required the number of sides.

II. *Problems in Geometric Progression.*

1. If in any geometric progression four terms be taken, so that as many are wanting between the first and second, as between the third and fourth, these four terms will be in

geometric progression. And if all but the first term in every successive m terms be omitted, the remaining terms will form a geometric progression. Determine its common ratio, and the number of terms.

2. In the series $\sqrt{\frac{8}{3}} - 4 + 12\sqrt{\frac{2}{3}} - \text{\&c.}$, determine the sixth term.

3. Given 6, the second term of a geometric progression, and 54, the fourth term; find the first term.

4. The fifth term of a geometric progression is 8 times the second, and the third term is 12; find the progression.

5. In every geometric progression, the first term is to the second, as the sum of all the terms diminished by the last is to the sum of all diminished by the first.

6. Find the sums of the following series:

$13 + 39 + 117 + \text{\&c. to } 7 \text{ terms.}$

$8 + 20 + 50 + \text{\&c. to } 15 \text{ terms.}$

$100 + 40 + 16 + \text{\&c. to } 10 \text{ terms.}$

$1 + \frac{2}{3} + \frac{4}{9} + \text{\&c. to } 10 \text{ terms.}$

$2 + 3 + \frac{9}{2} + \text{\&c. to } 20 \text{ terms.}$

$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \text{\&c. to } n \text{ terms.}$

$4 + 3 + \frac{9}{4} + \frac{27}{16} + \text{\&c. to } 10 \text{ terms.}$

$3 + 4\tfrac{1}{2} + 6\tfrac{3}{4} + \text{\&c. to } 5 \text{ terms.}$

$\frac{4}{9} + \frac{1}{3} + \frac{1}{4} + \frac{3}{16} + \text{\&c. to } 5 \text{ terms.}$

$\frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \text{\&c. to } n \text{ terms.}$

$1 - 2 + 4 - 8 + \text{\&c. to } n \text{ terms.}$

$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \text{\&c. to } n \text{ terms.}$

$8 - 6 + 4\frac{1}{2} -$ &c. to 10 terms, and *in inf.*

$21 - 3 + \frac{3}{7} -$ &c. to 6 terms.

$\frac{1}{3} - \frac{1}{2} + \quad \frac{3}{4} - \frac{9}{8} +$ &c. to n terms.

$\sqrt{2} + \sqrt{3} + \sqrt{4\frac{1}{2}} +$ &c. to 10 terms.

$\frac{2}{3} - \sqrt{\frac{2}{3}} + 1 -$ &c. to 8 terms.

$\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} +$ &c. to n terms.

$\sqrt{\frac{3}{5}} - \sqrt{6} + 2\sqrt{15} -$ &c. to 8 terms.

$3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} +$ &c. *in inf.*

$2 - \frac{1}{3} + \frac{1}{18} - \frac{1}{108} +$ &c. *in inf.*

$1 + \frac{1}{3} + \frac{1}{9} +$ &c. *in inf.*

$1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} +$ &c. *in inf.*

$\frac{2}{3} + \frac{2}{5} + \frac{6}{25} +$ &c. *in inf.*

$\frac{1}{3} - \frac{1}{6} + \frac{1}{12} - \frac{1}{24} +$ &c. *in inf.*

$\frac{1}{3} - \frac{1}{3.2} + \frac{1}{3.2^2} +$ &c. *in inf.*

$\frac{1}{2} - \frac{1}{2.2^3} + \frac{1}{2.2^6} -$ &c. *in inf.*

$\frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} +$ &c. *in inf.*

$1 + 1.1 + 1.01 + 1.001 +$ &c. to n terms.

$1 + \frac{2}{3} - \frac{1}{4} + \frac{3}{32} -$ &c. *in inf.*

$$\frac{a}{x}\sqrt{\frac{3}{2}}+\sqrt{\frac{a}{x}}+\sqrt{\frac{2x}{3a}}+\text{\&c. to } n \text{ terms and } \textit{inf.}$$

$$\sqrt[3]{\frac{9}{4}}+\frac{3}{2\sqrt[3]{2}}+\frac{3\sqrt[3]{3}}{4}+\text{\&c. to 9 terms.}$$

$$\frac{1}{3}+\frac{1}{6\sqrt{-1}}-\frac{1}{12}-\text{\&c. } \textit{in inf.}$$

$$\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{1}{2-\sqrt{2}}+\frac{1}{2}+\text{\&c. } \textit{in inf.}$$

$$r-\tfrac{1}{2}r^{\frac{3}{2}}+\tfrac{1}{4}r^{2}-\text{\&c. to } n \text{ terms.}$$

$$x^{\frac{3}{2}}-ax+\frac{a^{2}}{\sqrt{x}}-\text{\&c. to } n \text{ terms.}$$

$$\frac{1}{\sqrt{x}}-\frac{\sqrt{2}}{x}+\frac{2}{x\sqrt{x}}-\text{\&c. to } n \text{ terms.}$$

$$(a^{2}-b^{2})+(a+b)+\frac{a+b}{a-b}+\text{\&c. to } n \text{ terms.}$$

$$\frac{a+x}{a-x}+\frac{a-x}{a+x}+\left(\frac{a-x}{a+x}\right)^{3}+\text{\&c. to } n \text{ terms.}$$

$$1-x\sqrt{-1}-x^{2}+x^{3}\sqrt{-1}+\text{\&c. to } 2n \text{ terms.}$$

$$5\sqrt{x}+\frac{1}{\sqrt{x}}+3\sqrt{x}+\frac{1}{x}+\sqrt{x}+\frac{1}{x\sqrt{x}}+\text{\&c. to } n \text{ terms.}$$

$$a^{p+nq}+a^{p+(n-1).q}+\text{\&c. to } (n-1) \text{ terms.}$$

7. Which is greater of the two series, and by what quantities,

$$\left\{\begin{array}{l} \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\text{\&c. } \textit{in inf.} \\ \text{or } \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\text{\&c. } \textit{in inf.} \end{array}\right\}$$

$$\text{and also}\left\{\begin{array}{l} 2+1+\frac{1}{2}+\text{\&c. } \textit{in inf.} \\ \text{or } \frac{1}{6}+\frac{1}{3}+\frac{1}{2}+\text{\&c. to 12 terms.} \end{array}\right\};$$

and also $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} -$ &c. *in inf.*

or $\frac{1}{4} - \frac{1}{16} + \frac{1}{64} -$ &c. *in inf.*

8. Find an expression for the sum of n terms of the series $\frac{1}{5} - \frac{2}{15} + \frac{4}{45} -$ &c. that may be applied according as n is an even or odd number.

9. Show that $a - ar + ar^2 -$ &c. *in inf.* : $a + ar + ar^2 +$ &c. *in inf.* :: $1 - r : 1 + r$.

10. How many terms of the series $6 + 4 + 2\frac{2}{3} +$ &c. amount to 18?

11. If (a) and (b) be the two first terms of a decreasing geometric progression; prove that the sum of the series *in inf.* will be $= \frac{a^2}{a-b}$.

12. If n be the number of terms, l the last term, s the sum, and r the common ratio of a geometrical progression; show that $r^n - \frac{s}{s-l} \cdot r^{n-1} + \frac{l}{s-l} = 0$.

13. If a be the first, and l the last term of a geometric progression in which the common ratio is r, and the number of terms is n; then will the sum of the series be $=$

$$\frac{a^{\frac{n}{n-1}} \cdot (r^n - 1)}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}.$$

14. The sum of a geometric series continued *in inf.* is $= 3$, and the sum of its first two terms is $2\frac{2}{3}$; find the series.

15. If a be the first term of a geometric series, and b the sum of the first three terms; find the common ratio.

16. The first term of a geometric series continued *in inf.* $= 1$, and any term is equal to the sum of all the succeeding terms; determine the series.

17. If the common ratio of a geometric progression be less than $\frac{1}{2}$; prove that any term is greater than the sum of all that follow.

18. The second and third terms of a geometric series are together $= 24$, and the two next $= 216$; determine the first term.

19. Find the geometric progression when the sum of the first and second terms is 9, and the sum of the first and third is 15.

20. The sum of the first and third terms of a geometric series is a, and of the second and fourth is b; find the first term.

21. The $(n+3)^{\text{th}}$ term of a geometric series is equal to the sum of the first and $(n+1)^{\text{th}}$ terms, and the sum of the $(n+1)^{\text{th}}$ and $(n+3)^{\text{th}}$ terms is equal to 7 times the first term; required the first term when the sum of m terms $= \left(\frac{2}{\sqrt{3}}\right)^m - 1$.

22. If four quantities be in geometric progression, the sum of the two extremes is greater than the sum of the two means.

23. In any geometric progression the sum of the first and last terms is greater than the sum of any other two terms equidistant from the extremes.

24. If any quantities whose differences are inconsiderable in respect to the quantities themselves, be in arithmetic progression, the same quantities are also in geometric progression.

25. If $a, b, c, d \ldots\ldots$ be n quantities in geometric progression; then will $\frac{1}{a^2 - b^2}, \frac{1}{b^2 - c^2}, \frac{1}{c^2 - d^2} \ldots\ldots$ be in geometric progression, and the sum of n terms will be

$$\frac{1}{b^{2(n-1)}} \times \frac{a^{2n} - b^{2n}}{(a^2 - b^2)^2}.$$

26. If there be an arithmetic and geometric series, each containing an odd number of terms, and the middle term of

each be the same; the sum of any two terms of the arithmetic series equidistant from the middle term, multiplied by the middle term, is equal to twice the product of any two terms of the geometric series equidistant from the middle term.

27. The sum of a series of quantities in geometric progression wanting the first term, is equal to the sum of all the terms except the last, multiplied by the common ratio; required proof.

28. Required the sum of the first (p) terms of the series whose $(n)^{\text{th}}$ term is $na + a^n$.

29. If r and r^1 be the ratios of two geometric progressions whose first terms are 1; prove that the difference of the sums of n terms is divisible by $r - r^1$.

30. If the sum s of $2n$ terms of a geometric series, whose first term is a, and common ratio r, be equal to (x) times the $(n+1)^{\text{th}}$ term; show that $x = \dfrac{s}{\sqrt{a^2 + (r-1)sa}}$.

31. If S and s be the sums of two geometric series continued *in inf.*, and such that $Ss = 1$; and the first two terms of the former be 1 and $\frac{1}{3}$, and the second term of the latter $-\frac{1}{2}$; find the second series.

32. Having given the sum (S) and the sum of the squares (s^2) of a geometric series continued *in inf.*; show that the common ratio is $= \dfrac{S^2 - s^2}{S^2 + s^2}$, and the first term $= \dfrac{2Ss^2}{S^2 + s^2}$.

33. If S and S_1 be the sums to infinity and to n terms of a decreasing geometric progression whose first term is a; show that $n \,.\, \text{Log.}\left(1 - \dfrac{a}{S}\right) = \text{Log.}\left(1 - \dfrac{S_1}{S}\right)$.

34. If S and S_1 be respectively the sum of n terms of the series $a + ar + ar^2 + \ldots\ldots$, and $a + ar^{-1} + ar^{-2} + \ldots\ldots$; and if l be the last term of the first series, then will $aS = lS_1$.

35. Having given the first term and common ratio of

a geometric series, find the number of terms, when the sum is equal to m times the sum of the same series with its terms inverted.

36. In every geometric progression consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the excess of the odd terms above the even.

37. If P be the product, S the sum, and s the sum of the reciprocals of n quantities in geometric progression; prove that $P^2 = \left(\frac{S}{s}\right)^n$.

38. Find four geometric means between 1 and 32; and two between 1 and 100; and three between $\frac{1}{9}$ and 9; and three between 2 and $\frac{1}{8}$.

39. The sum and difference of the arithmetic and geometric means between two numbers are 9 and 1; find them.

40. The arithmetic mean between two quantities is to the geometric mean :: 5 : 3; it is required to find the ratio of the quantities themselves.

41. The difference between two numbers is 12, and the arithmetic is to the geometric mean :: 5 : 3; determine the numbers.

42. If the arithmetic mean between (a) and (b) is double the geometric; prove that $a : b :: 2 + \sqrt{3} : 2 - \sqrt{3}$.

43. Between $n + 1$ quantities (x, y), $(x, 2y)$, $(x, 4y)$, are inserted n geometric means, and M_1, M_2, M_3, &c. are the n^{th} terms respectively; prove that

$$\frac{M_1}{M_2} + \frac{M_2}{M_3} + \frac{M_3}{M_4} + \ldots\ldots = \sqrt[n+1]{\frac{n^{n+1}}{2^n}}.$$

44. If the ratio of the geometric to the arithmetic mean between two quantities x and y be as $1 : n$; prove that

$$x : y :: \sqrt{n^2 + 1} + \sqrt{n^2 - 1} : \sqrt{n^2 + 1} - \sqrt{n^2 - 1}.$$

45. In a geometric series, having given (a) and (b) the $(p)^{th}$ and $(q)^{th}$ terms, determine the value of the $(x)^{th}$ term, and the sum of x terms.

46. Having given the same; determine the value of the $(p+q)^{th}$ term.

47. Having given the sum of the n^{th} and $2n^{th}$ terms of a geometric series $=p$, and the sum of the $2n^{th}$ and $3n^{th}$ terms $=q$; determine the first term and common ratio.

48. In a geometric series, if the $(p+q)^{th}$ term $=m$, and the $(p-q)^{th}$ term $=n$; prove that the $(p)^{th}$ term

$$=\sqrt{mn}, \text{ and the } (q)^{th} \text{ term} = m\left(\frac{n}{m}\right)^{\frac{p}{2q}}.$$

49. If x, y, z be the p^{th}, q^{th}, r^{th} terms of a geometric progression; then will $x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = 1$.

50. There are two geometric series S and S' continued *in inf.*, and $S : S' :: 4 : 9$, and the two first terms of S are 40 and 35; the second term of the second series being $46\frac{19}{36}$; determine the first term and the ratio of the second series.

51. If S be the sum of n terms of a geometrical progression, whose first term is x_1, and $x_2, x_3, \ldots x_n$ denote the sums of the first two, three, four, &c. terms; then will

$$(S+x_1)+(S+x_2)+\&\text{c.} = \frac{x_1}{r-1} \cdot \left\{n \cdot (r^n - 2) + \frac{r}{r-1} \cdot (r^n - 1)\right\}.$$

52. If $S = (x-y) + \left(y - \frac{y^2}{x}\right) + \&\text{c. } in\ inf.$, and S' the sum of n terms of the same series; show that

$$S : S' :: x^n : x^n - y^n.$$

53. If $S = (x-y) + \left(\frac{y^2}{x} - \frac{y^3}{x^2}\right) + \&\text{c. to } n$ terms, and $\Sigma =$ the sum of the same series continued *in inf.*; prove that $S : \Sigma :: x^{2n} - y^{2n} : x^{2n}$.

54. If there be n quantities forming a geometric progression whose common ratio is r, and S_m denote the sum of the first m terms of such a series; prove that the sum of their products taken two and two together $= \frac{r}{r+1} S_n S_{n-1}$.

55. If S_m, S'_m denote the sums of m terms of geometric progressions whose common ratios are r and r' respectively; find the sum of n terms of the series $SS'_1 + S_2S'_2 + S_3S'_3 +$ &c.

56. If $S_1, S_2, S_3, \ldots. S_n$ be the sums of n geometric progressions whose first terms are $a, 2a, 3a \ldots\ldots na$; then will

$$S_1 + S_2 + S_3 \ldots\ldots + S_n = \frac{n.(n+1)}{2} \cdot \frac{r^n - 1}{r-1} \cdot a.$$

57. In a geometric progression, having given the first term and common ratio, find the sum of the series

$$S_n + S_{2n} + S_{3n} + \text{\&c. to } r \text{ terms.}$$

58. If $S, S_1, S_2, S_3 \ldots\ldots$ be the sums of any number of geometric series continued *in inf.*, and having their common ratios in arithmetical progression; determine the sum of $\frac{1}{S} + \frac{1}{S_1} + \frac{1}{S_2} +$ &c. to n terms.

59. If $S_1, S_2, S_3, \ldots\ldots S_n$ be the sums of n geometric series continued *in inf.*, the first term of which is 1, and the common ratio $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \ldots \frac{1}{r^n}$ respectively; determine the sum of the reciprocals $\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} \ldots\ldots + \frac{1}{S_n}$.

60. If there be an infinite number of decreasing geometric series, each continued *in inf.*; a, a^2, a^3, &c. the first terms; $r, 2r, 3r$, &c. the common ratios; S_1, S_2, S_3, &c. the sums; prove that $\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \text{\&c.} = \frac{a.(1-r)-1}{(a-1)^2}$.

61. If there be an infinite number of geometric progressions

wherein the first term of each is the n^{th} term of that which immediately precedes it; then will their sum be

$$\frac{a}{(1-r).(1-r^{n-1})}.$$

62. Find the sum of an infinite series of quantities

$$\frac{a}{c}+\frac{a+b}{ce}+\frac{a+2b}{ce^2}+\frac{a+3b}{ce^3}+\&c.,$$

$$\text{and } \frac{a}{\beta}-\frac{a+\gamma}{\beta^2}+\frac{a+2\gamma}{\beta^3}-\&c. \text{ to } n \text{ terms,}$$

of which the numerators are in arithmetic progression, and the denominators in geometric. Also the sum of

$$\frac{x}{r}+\frac{x+y}{r^{p+1}}+\frac{x+2y}{r^{2p+1}}+\&c. \text{ to } n \text{ terms and } in\ inf.,$$

$$\text{and } \frac{1}{2}+\frac{5}{4}+\frac{9}{8}+\frac{13}{16}+\&c. \ in\ inf.$$

63. Given $\frac{2}{3}, \frac{5}{7}$ the first two terms of an arithmetic progression; find the sum of 15 terms. And if the same quantities be the first two terms of a geometrical progression; find the sum of 15 terms.

64. If the terms of an arithmetic progression, a, $a+r$, $a+2r$, &c. be multiplied by the corresponding terms of a geometric progression, b, bd, bd^2, &c. of the same number of terms; determine the sum of the resulting series.

65. Determine the sum of $7+5+$ &c. an arithmetic series of 12 terms; also of $-\frac{7}{5}+\frac{7}{2}-$ &c. a geometric series of 5 terms: and also of the infinite series, whose terms are the products of the corresponding terms of these series continued *in inf.*

66. There are two infinite geometric progressions, each beginning from 1, whose sums are σ and σ^1; prove that the

sum of the series formed by multiplying their corresponding terms is $\frac{\sigma\sigma^1}{\sigma + \sigma^1 - 1}$.

67. If σ_1 represent the sum of a geometric progression continued *in inf.*, σ_2 the sum of the squares of the terms, σ_3 the sum of their cubes, &c.; then will

$$\frac{1}{\sigma_1} \pm \frac{1}{\sigma_2} + \frac{1}{\sigma_3} \pm \text{\&c. } in\ inf. = \frac{1}{a \mp 1} - \frac{r}{a \mp r}.$$

68. If $S_1 = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \text{\&c. } in\ inf.$

and $S_2 = 1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \text{\&c. } in\ inf.$

prove that $S_1 : S_2 :: 27 : 1$.

69. If $S_1 = 1 + \frac{1}{r} + \frac{1}{r^2} + \text{\&c. } in\ inf.$

$$S_2 = 1 - \frac{1}{r} + \frac{1}{r^2} - \text{\&c.} \ldots\ldots$$

$$S_3 = 1 + \frac{1}{r^2} + \frac{1}{r^4} + \frac{1}{r^6} \ldots\ldots$$

prove that $S_1 \times S_2 = S_3$.

70. Determine the sum of the series

$$ar + 3ar^2 + 6ar^3 + 10ar^4 + \text{\&c.}$$

which arises from multiplying the terms of a geometric progression by the corresponding terms of a series of triangular numbers.

71. Find the sums of the following series:

$1 + 2x + 3x^2 + \text{\&c.}$ to n terms.

$1 + \frac{2}{x} + \frac{3}{x^2} + \frac{4}{x^3} + \text{\&c. } in\ inf.$

$1 - \frac{2x^2}{a^2} + \frac{3x^4}{a^4} - \frac{4x^6}{a^6} + \text{\&c. } in\ inf.$

$1.2.x + 2.3.x^2 + 3.4.x^3 +$ &c. *in inf.* and to n terms.
$1.2.3.x + 2.3.4.x^2 + 3.4.5.x^3 +$ &c. *in inf.* and to n terms.
$1.2.3.4.x + 2.3.4.5.x^2 + 3.4.5.6.x^3 +$ &c. *in inf.* & to n terms.
$1.x + 3x^2 + 5x^3 + 7x^4 +$ &c. *in inf.*
$1.x + 5x^2 + 9x^3 +$ &c. *in inf.*
$1.2.x + 3.4.x^2 + 5.6.x^3 +$ &c. *in inf.*
$1.2.3.x + 4.5.6.x^2 + 7.8.9.x^3 +$ &c. *in inf.*
$1.3.x + 2.4.x^2 + 3.5.x^3 +$ &c. *in inf.*
$1.4.x + 2.5.x^2 + 3.6.x^3 +$ &c. *in inf.*
$1.4.x + 3.6.x^2 + 5.8.x^3 +$ &c. *in inf.*
$1.3.x + 4.6.x^2 + 7.9.x^3 +$ &c. *in inf.*
$1.3.x + 3.5.x^2 + 5.7.x^3 +$ &c. *in inf.*

72. Prove that the sum of n terms of the series $1^3 + 3^3 + 5^3 + 7^3 +$ &c. is a hexagonal number whose root is n^2.

73. Prove that the sum of the reciprocals of the n^{th} powers of the odd numbers is to the sum of the reciprocals of the same powers of the even numbers :: $1 : 2^n - 1$.

74. Reduce to its lowest terms $\dfrac{.3242424\ldots.}{1 - \frac{2}{3} + \frac{4}{9} - \text{\&c.}}$,

" " $\dfrac{1.666\ldots..}{1 - \frac{3}{5} + \frac{9}{25} - \text{\&c.}}$,

" " $\dfrac{.02777\ldots..}{\frac{1}{3} - \frac{1}{2} + \frac{3}{4} - \text{\&c.}}$.

III. *Problems in Harmonic Progression.*

1. EXPLAIN the nature of harmonic progression; and continue in both directions the series 2, 3, 6.

2. Continue the harmonic progression .... 3, 4, 6 .... upwards and downwards. How far can it be continued either way?

3. Continue to 3 terms each way the series 1, $1\frac{1}{6}$, $1\frac{2}{5}$.

4. If a and b be the first two terms of a series in harmonic progression, continue the series to three more terms, and find the n^{th} term.

5. Prove that the reciprocals of quantities in harmonic progression are in arithmetic progression.

6. In any harmonic progression, the product of the first two terms is to the product of any two adjacent terms as the difference between the two first is to the difference between the two others.

7. In any harmonic progression, the difference between the first two terms is to the difference between any two others as the second term diminished by (n) times the difference between the first and second is to the last; where $n =$ the number of terms between the first and last.

8. In any harmonic progression, the second term diminished by (n) times the difference between the first and second is to the last as the product of the two first is to the product of the two last; n as before.

9. Any term of an harmonic progression is equal to the product of the first two terms divided by the difference between [the second] and [n times the difference between the first and second].

10. The sum of any two terms of an harmonic progression is greater than twice the intermediate mean term; and this excess is greater, as they are the more remote.

11. Find a fourth harmonic proportional to 6, 8, 12.

12. Find the sum of n terms of the series $\frac{1}{n}, \frac{1}{m}, \frac{2n-m}{mn}, \frac{3n-2m}{mn}$, &c.

13. If the two extremes and the number of terms in an harmonic progression be known; the intervening series may be found.

14. Insert two harmonic means between 2 and 4; two between 6 and 24; four between 2 and 12; six between 1 and 20; six between 3 and $\frac{6}{23}$; and seven between 10 and 12.

15. Find two numbers whose difference is 8, and the harmonic mean between them $1\frac{4}{5}$.

16. Insert n harmonic means between a and b; and if a_1 be the first harmonic mean, prove that

$$a : b :: a^{n+1} : a_1^{n+1}.$$

17. The difference between two numbers is 18, and 4 times the geometric mean is equal to 5 times the harmonic mean; find the numbers.

18. Prove that a geometric mean between two quantities is a mean proportional between an arithmetic and harmonic mean between the same two quantities.

19. If (a) be an arithmetic, (b) a geometric, and (c) an harmonic mean between two quantities; show that a is greater than b, and b greater than c.

20. The difference of the arithmetic and harmonic means between two numbers is $1\frac{4}{5}$; find the numbers, one being 4 times the other.

21. The arithmetic mean between two numbers exceeds the geometric by 13, and the geometric exceeds the harmonic by 12; what are the numbers?

22. If the arithmetic mean between two quantities x and y be m times the harmonic mean; then will

$$x : y :: \sqrt{m} + \sqrt{m - 1} : \sqrt{m} - \sqrt{m - 1}.$$

23. If the geometric mean between two quantities x and y, be to the harmonic as $1 : n$; show that

$$x : y :: 1 + \sqrt{(1 - n^2)} : 1 - \sqrt{(1 - n^2)}.$$

24. If m harmonic means be inserted between a and pa; prove that the ratio of the first : the last is = to the ratio of $m + p : mp + 1$.

25. If y be an harmonic mean between x and z, and x and z be respectively the arithmetic and geometric means between a and b, show that

$$y = \frac{2 \cdot (a+b)}{\left\{\left(\frac{a}{b}\right)^{\frac{1}{4}} + \left(\frac{b}{a}\right)^{\frac{1}{4}}\right\}^2}$$

26. Having given (a) the sum of three numbers in harmonic progression, and (b) their continual product; determine the numbers.

27. There are three numbers in harmonic progression; if 1 be subtracted from the first, the progression becomes geometric; and if 4 be subtracted from the third, it becomes arithmetic. What are the numbers?

28. From each of three quantities in harmonic progression, what quantity must be subtracted that the three results may be in geometric progression?

29. There are four numbers, the first three of which are in arithmetic and the last three in harmonic progression; prove that the first has to the second the same ratio which the third has to the fourth.

30. The sum of three consecutive terms of an harmonic progression, whose first term is $\frac{1}{2}$, is $= 1\frac{1}{12}$; determine the progression, and continue it both ways.

31. If S and s be the sums of two infinite series, the common ratios of whose terms are R and r respectively; then S, s, R, r are in harmonical progression, the form of each series being (r, r^2, r^3, &c.) and r and R fractional.

32. Given M and N, the m^{th} and n^{th} terms of an harmonic progression; find the first and second terms.

33. Having given the two first terms of an harmonic progression; determine the $(n)^{th}$ term.

34. Having given the $(m)^{th}$ and $(n)^{th}$ terms of an harmonic progression; determine the $(m+n)^{th}$ term.

35. If a, b, c be the p^{th}, q^{th}, r^{th} terms of an harmonic progression; then will $(p-q) \,.\, ab + (r-p) \,.\, ac + (q-r) \,.\, bc = 0$.

36. If $a^x = b^y = c^z =$ &c., and a, b, c, &c. be in geometric progression; then will x, y, z, &c. be in harmonic progression.

37. Compare the lengths of the sides of a right-angled triangle, when the squares described upon them are in harmonic progression.

APPENDIX II.

1. Form the equation, whose roots are

$$2 + \sqrt{-3},\ 2 - \sqrt{-3},\ 1, \text{ and } -5.$$

2. Also whose roots are $\pm\sqrt{-2}$, 3, and 4.

3. Also whose roots are $1 \pm \sqrt{-2}$, and $2 \pm \sqrt{-3}$.

4. Form the biquadratic equation, two of whose roots are

$$1 + \sqrt{a^3}, \text{ and } -\sqrt{-b}.$$

5. Also if two of the roots be $\sqrt{3}$, and $-\sqrt{-5}$.

6. Determine the equation whose roots are

$$\frac{1}{2}a + \sqrt{\left(-\frac{3}{4}a^2\right)},\ \frac{1}{2}a - \sqrt{\left(-\frac{3}{4}a^2\right)}, \text{ and } -a.$$

7. Form the equation, of which the roots are the different values of $a + \sqrt[n]{b}$.

8. Given that an equation has *one* root; show that it will have as many roots as it has dimensions.

9. If any coefficient in an equation be changed; prove that all the roots will be changed.

10. If a be a root of the equation

$$x^n + px^{n-1} \ldots\ldots + Px^3 + Qx^2 + Rx + S = 0,$$

and if $\frac{S}{a} + R = R_1$, $\frac{R_1}{a} + Q = Q_1$, &c.;

show that $\frac{S}{a}$, R_1, Q_1, &c. are integers.

11. If α, β, γ, &c. be the roots of the equation

$$x^n + px^{n-1} \ldots\ldots + Qx + P = 0;$$

show that

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \&\text{c.} = \frac{pQ - nP}{P}.$$

12. The roots of the equation

$x^n - px^{n-1} + qx^{n-2} \ldots\ldots - Qx + R = 0$, being α, β, γ, &c.;

show that

$$\frac{\alpha^2}{\beta} + \frac{\alpha^2}{\gamma} + \&\text{c.} + \frac{\beta^2}{\alpha} + \frac{\beta^2}{\gamma} + \&\text{c.} + \frac{\gamma^2}{\alpha} + \frac{\gamma^2}{\beta} + \&\text{c.}$$

$$= (p^2 - 2q) \cdot \frac{Q}{R} - p.$$

13. If $x^n - px^{n-1} + qx^{n-2} - \&\text{c.} \pm W = A$, and a be any root of the equation $x^n - px^{n-1} + qx^{n-2} - \&\text{c.} \pm W = 0$, prove that $x - a$ is a divisor of the expression

$$x^n - px^{n-1} + qx^{n-2} - \&\text{c.} \pm W.$$

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14. Take away the second term of the following equations:

1. $x^3 - 9x^2 + 26x - 34 = 0.$
2. $x^3 - 3x^2 + 4x - 5 = 0.$
3. $x^4 + 24x^3 - 12x^2 + 4x - 30 = 0.$
4. $x^4 + 8x^3 + x^2 - x - 10 = 0.$

15. Take away the third term of the following equations:

1. $x^3 - 6x^2 + 9x - 20 = 0.$
2. $x^3 - 4x^2 + 5x - 2 = 0.$

16. Prove that the third term of the equation

$$x^3 - px^2 + qx - r = 0,$$

cannot be taken away by the common method, if p^2 be less than $3q$. Show how it *may* be taken away in this case.

17. In an equation of n dimensions, show that the second and third terms may be taken away by the same transformation, when the square of the sum of the roots is to the sum of their squares $:: n : 1$.

18. Exterminate the last term but one of an equation of five dimensions by the solution of a simple equation.

19. Transform the equation $x^3 - \dfrac{ax^2}{m} + \dfrac{bx}{n} + \dfrac{c}{p} = 0$, into one whose coefficients shall be integral.

20. Transform the equation $y^3 - 2py^2 - 33p^2y + 14p^3 = 0$, into one whose coefficients shall be numerical.

21. Transform the equation

$$x^n - a^{\frac{1}{2}}px^{n-1} + qx^{n-2} - a^{\frac{1}{2}}rx^{n-3} + \&c. = 0,$$

into one whose coefficients are rational.

22. Transform the following equations into others whose terms shall be alternately positive and negative:

1. $x^3 - x^2 + \dfrac{3x}{2} + 3 = 0.$
2. $x^4 + x^3 - 19x^2 + 11x + 30 = 0.$

23. Transform the equation $x^3 + x^2 - 10x + 4 = 0$, into one whose roots shall be greater by 4 than the roots of the given equation.

24. Transform the equation $x^4 - 4x^3 + 6x^2 - 12 = 0$, into one whose roots shall be greater by 5 than the roots of the given equation.

25. Transform the equation $x^3 - 6x^2 + 9x - 12 = 0$, into one whose roots shall be less by 6 than the roots of the given equation.

26. Transform the equation $3x^3 - 12x^2 + 15x - 21 = 0$, into one whose roots shall be treble the roots of the given equation.

27. Transform the equation $x^4 - 2x^2 - 3x + 4 = 0$, into one whose roots shall be one-eighth of the roots of the given equation.

28. If a, b, c, &c. be the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \text{\&c.} = 0;$$

transform it into one whose roots are ma, mb, mc, &c.

29. If the roots of the equation $x^3 - px^2 + qx - r = 0$, be a, b, c; transform it into another whose roots shall be

$$a + b,\ a + c,\ b + c.$$

30. Transform the equation $x^4 - 40x + 39 = 0$, into one whose roots shall be the sum of every two roots of the original equation.

31. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into another whose roots are

$$\frac{1}{a+b},\ \frac{1}{a+c},\ \frac{1}{b+c}.$$

32. If the roots of the equation

$$x^3 - px^2 + qx - r = 0, \text{ be } a,\ b,\ c;$$

determine the equation whose roots are ab, ac, bc.

33. Transform the equation $x^3 - px^2 + qx - r = 0$, into one whose roots shall be mean proportionals between the roots of the equation, and a given quantity (m).

34. If the roots of the equation $x^2 - px + q = 0$, be a and b; determine the equation, of which $-\sqrt{a}$, and $-\sqrt{b}$ are roots.

35. If the roots of the equation

$$x^2 - px + q = 0, \text{ be } a \text{ and } b;$$

determine the equation whose roots are an arithmetic, a geometric, and an harmonic mean between a and b.

36. Transform the equation $x^3 - 2x^2 + 2x - 4 = 0$, into one whose roots are the squares of the roots of the original equation.

37. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into one whose roots are a^2, b^2, c^2.

38. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into one whose roots are $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$.

39. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into one whose roots are

$$a^2 + b^2,\ a^2 + c^2,\ b^2 + c^2.$$

40. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into one whose roots are

$$\frac{1}{a^2} + \frac{1}{b^2},\ \frac{1}{a^2} + \frac{1}{c^2},\ \frac{1}{b^2} + \frac{1}{c^2}.$$

41. Transform the equation $x^3 - 6x^2 + 11x - 6 = 0$, into one whose roots are $\frac{1}{a^2 + b^2}$, $\frac{1}{a^2 + c^2}$, and $\frac{1}{b^2 + c^2}$.

42. Transform the equation $x^4 + x^3 + x^2 + x + 1 = 0$, into one whose roots shall be the squares of the roots of the given equation; and show from the roots themselves that the transformation is correct.

43. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into one whose roots are

$$\left(\frac{a}{b} + \frac{b}{a}\right),\ \left(\frac{a}{c} + \frac{c}{a}\right),\ \text{and}\ \left(\frac{b}{c} + \frac{c}{b}\right).$$

44. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into one whose roots are

$$\frac{c}{a + b - c},\ \frac{b}{a + c - b},\ \frac{a}{b + c - a}.$$

45. Transform the equation

$$x^3 - px^2 + qx - r = 0,$$

whose roots are a, b, c, into another whose roots are

$$a + b + ab,\ a + c + ac,\ b + c + bc.$$

46. Transform an equation into one whose roots shall be the squares of the differences of the roots of the original equation; and show by means of this transformation how the number of impossible roots in an equation of five dimensions may be detected.

47. Transform the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, into one whose roots are the reciprocals of every $(n - 1)$ roots of the original equation.

48. If the roots of the equation $x^3 - px^2 + qx - r = 0$, be a, b, c; transform it into one whose roots are a^3, b^3, c^3.

49. If $x^3 - 2x^2 + 1 = 0$; deduce the equation of which the roots are the cubes of the roots of the original equation.

50. Transform the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, into one whose m^{th} term shall be a given quantity.

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51. Determine the roots of the equation

$$x^4 - 4x^3\sqrt[4]{2} + 6x^2\sqrt[4]{4} - 4x\sqrt[4]{8} + 2 = 0.$$

52. Solve the equation $x^3 - 4x^2 - 3x + 12 = 0$, one root of which is of the form $\sqrt{a}$.

53. One root of the equation $x^3 - 6x^2 + 6x + 8 = 0$, being $1 + \sqrt{3}$; find all the roots.

54. One root of the equation $x^3 - 11x^2 + 37x - 35 = 0$, being $3 + \sqrt{2}$; find all the roots.

55. One root of the equation $x^4 + x^3 - 8x^2 - 16x - 8 = 0$, being $1 - \sqrt{5}$; find all the roots.

56. Solve the following equations, two of whose roots are equal:

1. $x^3 - 7x^2 + 16x - 12 = 0.$
2. $x^3 + 8x^2 + 20x + 16 = 0.$
3. $x^3 - 5x^2 + 8x - 4 = 0.$
4. $x^3 - 5x^2 - 8x + 48 = 0.$
5. $x^3 - x^2 - 8x + 12 = 0.$
6. $x^4 - \frac{1}{2}x + \frac{3}{16} = 0.$
7. $x^3 + \frac{10}{7}x^2 - \frac{4000}{9261} = 0.$

57. The equation $3x^5 - 10x^3 + 15x + 8 = 0$, has three equal roots; determine them.

58. Solve the equation $x^4 - 14x^3 + 61x^2 - 84x + 36 = 0$, whose roots are of the form a, a, b, b.

59. Solve the equation

$$x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0,$$

whose roots are of the form a, a, a, b, b.

60. The equation $x^6 - 2x^5 + 6x^4 - 8x^3 + 12x^2 - 8x + 8 = 0$, has equal roots; determine them.

61. Solve the equation $x^4 + px^3 + qx^2 + rx + s = 0$, which has two pairs of equal roots.

62. Solve the following equations, which have two roots of the form $+a, -a$,

1. $4x^3 - 32x^2 - x + 8 = 0.$
2. $x^4 - 5x^3 - 5x^2 + 45x - 36 = 0.$
3. $x^4 + 3x^3 - 7x^2 - 27x - 18 = 0.$
4. $x^4 + x^3 - 11x^2 + 9x + 18 = 0.$

63. Solve the equation

$$x^5 - 10x^4 + 29x^3 - 10x^2 - 62x + 60 = 0,$$

two of its roots being 3 and $\sqrt{2}$.

64. The equation $x^3 - 15x^2 + 66x - 80 = 0$, has two roots whose sum is 13; find all the roots.

65. The equation $x^4 - 45x^2 - 40x + 84 = 0$, has two roots whose difference is 3; determine all the roots.

66. The roots of the equation $x^3 - 15x^2 + 66x - 80 = 0$, have a common difference; determine them.

67. In the equation $x^3 - 6x^2 + 11x - 6 = 0$, one root is double another; determine all the roots.

68. The product of two roots of the equation

$$x^4 + x^3 - 62x^2 - 80x + 1200 = 0, \text{ is } 30;$$

determine all the roots.

69. Determine the roots of the equation

$$x^3 - 17x^2 + 94x - 168 = 0,$$

two of them being in the proportion of 2 : 3.

70. In the equation $x^3 - 10x^2 + 27x - 18 = 0$, the greatest root is double of the second, and the second treble of the third; determine all the roots.

71. One root of the equation $x^4 - 5x^3 - x + 5 = 0$ is 5; determine all the roots.

72. The equation $x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0$, has two roots of the form a, $\frac{1}{a}$; determine all the roots.

73. The roots of the equation

$$6x^4 - 43x^3 + 107x^2 - 108x + 36 = 0,$$

are of the form a, b, $\frac{b}{a}$ and $\frac{a}{b}$; determine them.

74. Determine the roots of the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0,$$

they being of the form $a + 1$, $a - 1$, $b + 1$, $b - 1$.

75. Solve the following equations, whose roots are in arithmetical progression:

1. $x^3 - 6x^2 - 4x + 24 = 0$.
2. $x^3 - 9x^2 + 23x - 16 = 0$.
3. $x^3 - 6x^2 + 11x - 6 = 0$.
4. $x^3 - 3x^2 + 6x + 8 = 0$.
5. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.
6. $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$.
7. $x^4 + x^3 - 11x^2 + 9x + 18 = 0$.

76. The roots of the equation $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$ being in arithmetical progression; prove that the least root is

$$\frac{p}{n} - \frac{n-1}{n}\sqrt{\left\{\frac{(n-1).3p^2 - 6nq}{n^2 - 1}\right\}},$$

and the common difference

$$\frac{2}{n}.\sqrt{\left\{\frac{(n-1).3p^2 - 6nq}{n^2 - 1}\right\}}.$$

77. Solve the following equations, whose roots are in geometrical progression:

1. $x^3 - 7x^2 + 14x - 8 = 0$.
2. $x^3 - 13x^2 + 39x - 27 = 0$.
3. $x^3 - 14x^2 + 56x - 64 = 0$.
4. $x^3 - 26x^2 + 156x - 216 = 0$.
5. $x^3 - px^2 + qx - r = 0$.
6. $x^4 + px^3 + qx^2 + rx + s = 0$.

78. If the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \text{\&c.} = 0,$$

be in geometrical progression; having given $p = 15$, $q = 70$; find n, r, &c.

79. Solve the following equations, whose roots are in harmonical progression:

1. $x^3 - 11x^2 + 36x - 36 = 0.$
2. $x^3 - 13x^2 + 54x - 72 = 0.$
3. $x^3 - \frac{5}{2}x^2 + x - \frac{1}{6} = 0.$
4. $8x^3 - 6x^2 - 3x + 1 = 0.$
5. $x^3 - \frac{13}{12}x^2 + \frac{9}{24}x - \frac{1}{24} = 0.$
6. $ax^3 - bx^2 - cx + 1 = 0.$

80. In the common cubic $x^3 - px^2 + qx - r = 0$, if the roots are in harmonic progression, and p, q, r, integer numbers, then r is the square of the greatest root. Apply this to solve the equation $x^3 - 23x^2 + 135x - 225 = 0.$

81. If the roots of the equation $x^3 - px^2 + qx - r = 0$, be in harmonic progression; show that

$$x^2 - \frac{pq - 3r}{q} \cdot x + \frac{p^2 - 3pqr + 9r^2}{q^2} = 0,$$

contains the greatest and least.

82. If the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \ldots\ldots + Qx^2 - Px + L = 0,$$

be in harmonic progression, then will the greatest and least be

$$\frac{n\sqrt{(n+1)} \cdot L}{\sqrt{(n+1)} \cdot P - \sqrt{\{3 \cdot (n-1)^2 \cdot P^2 - 6n \cdot (n-1) \cdot QL\}}},$$

and $$\frac{n \cdot \sqrt{(n+1)} \cdot L}{\sqrt{(n+1)} \cdot P + \sqrt{\{3 \cdot (n-1)^2 \cdot P^2 - 6n \cdot (n-1) \cdot QL\}}}.$$

83. Solve the equation $x^3 - 31x^2 + 300x - 900 = 0$, whose roots are successive triangular numbers.

84. Explain the method of finding the equal roots of equations, and apply it to the equations

1. $x^4 - 9x^2 + 4x + 12 = 0.$
2. $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0.$

85. Having given the equation

$$2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0,$$

determine whether it has equal roots.

86. If the equation $x^5 + qx^3 - rx^2 - t = 0$ has two roots equal to each other; prove that one of them will be a root of the quadratic $x^2 + \frac{2q^2}{5r} \cdot x + \frac{5t}{3r} - \frac{4q}{15} = 0$.

87. Show that if an equation have two equal roots, and the terms be multiplied by the terms of an arithmetic progression, the result will $= 0$.

88. If an equation have (n) equal roots, the equation formed by multiplying the terms by the terms of an arithmetic progression has $(n - 1)$ of them.

89. Having given

$$\left.\begin{aligned} x^3 - px^2 + qx - r &= 0, \text{ whose roots are } a, b, c; \\ \text{and } x^3 - p^1x^2 + q^1x - r^1 &= 0, \text{ whose roots are } a, b, c^1; \end{aligned}\right\}$$

find c, and c^1.

90. One root being common to the two equations

$$x^3 - 9x^2 + 26x - 24 = 0,$$
$$\text{and } x^3 - 7x^2 + 7x + 15 = 0,$$

find the remaining roots of each.

91. Determine all the roots of the two equations which have one common root

$$x^3 - 3x^2 + 11x - 9 = 0,$$
$$x^3 - 5x^2 + 11x - 7 = 0.$$

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92. Solve the equation $x^3 - 1 = 0$; and show that its roots are of the form a, b, b^{-1}.

93. The roots of the equation $x^4 + px^2 + 1 = 0$, must be of the form $a, b, \frac{1}{a}, \frac{1}{b}$; exhibit them in that form.

94. Solve the following recurring equations:

1. $x^4 - 3x^3 + 2x^2 - 3x + 1 = 0.$
2. $x^4 + 6ax^3 - 20a^2x^2 - 6a^3x + a^4 = 0.$
3. $x^4 - \frac{5}{2}x^3 + 2x^2 - \frac{5}{2}x + 1 = 0.$
4. $x^4 \pm 1 = 0.$
5. $x^5 - 21x^4 + 37x^3 - 37x^2 + 21x - 1 = 0.$
6. $x^5 - \frac{15}{2}x^4 + \frac{37}{2}x^3 - \frac{37}{2}x^2 + \frac{15}{2}x - 1 = 0.$
7. $x^5 + 4ax^4 - 12a^2x^3 - 12a^3x^2 + 4a^4x + a^5 = 0.$
8. $x^{12} - a^{12} = 0.$

95. Reduce

$x^9 - px^8 + qx^7 - rx^6 + sx^5 - sx^4 + rx^3 - qx^2 + px - 1 = 0,$

to an equation of four dimensions.

96. Exhibit the quadratic factors of the equation

$$x^m \pm 1 = 0.$$

97. Show that when m is a prime number, the roots both real and imaginary of the equation $x^m + 1 = 0$, are different powers of any one of its roots.

98. In any recurring equation $x^n - px^{n-1} + \&c. = 0$, whose roots are a, b, c, &c.; prove that

$$\frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{a^2} + \&c. = (p^2 - 2q + \sqrt{n}) \cdot (p^2 - 2q - \sqrt{n}).$$

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99. The roots of the cubic equation $x^3 - qx + r = 0$, are real when $\frac{q^3}{27}$ exceeds $\frac{r^2}{4}$.

100. The solution of the cubic equation $x^3 + qx + r = 0$ is dependent on the solution of the equation $x^3 - 1 = 0$.

101. Having given, 1, α, β, the cube roots of 1,

$$\text{and } A = \left\{-\frac{r}{2} + \left(\frac{r^2}{4} - \frac{q^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}},$$

$$\text{and } B = \left\{-\frac{r}{2} - \left(\frac{r^2}{4} - \frac{q^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}},$$

prove synthetically that $A + B$, $\alpha A + \beta B$, and $\beta A + \alpha B$ are the three roots of the equation.

102. If two roots of the cubic equation

$x^3 - qx + r = 0$ be $a + b\sqrt{-3}$, and $a - b\sqrt{-3}$;

$$\text{then will } -\frac{r}{2} + \sqrt{\left(\frac{r^2}{4} - \frac{q^3}{27}\right)} = (b - a)^3.$$

103. Explain in what case, and for what reason, Cardan's formula for the solution of a cubic equation does not enable us to determine the roots.

104. Determine whether Cardan's rule is applicable to the solution of the equation $x^3 - 237x - 884 = 0$.

105. Show that Cardan's rule for the solution of a cubic equation is applicable when all the roots are possible, and two of them equal; and by means of it, find the roots of the equation $x^3 + 6x^2 - 32 = 0$.

106. Solve the following equations by Cardan's rule:

1. $x^3 - 9x - 14 = 0$.
2. $x^3 - 6x - 40 = 0$.
3. $x^3 - 9x + 28 = 0$.
4. $x^3 + 3x^2 + 9x - 13 = 0$.
5. $x^3 - 6x^2 + 3x - 18 = 0$.
6. $x^3 - 12x^2 + 57x - 94 = 0$.
7. $x^3 + 6x^2 + 20x + 15 = 0$.
8. $x^3 - 12x^2 + 36x - 7 = 0$.
9. $x^{3m} - px^{2m} + qx^m - r = 0$.

107. Find by the doctrine of permutations, the roots of the equation $x^3 - qx + r = 0$.

108. Assuming the quadratic factors in Des Cartes's solution of a biquadratic to be $x^2 + ax + b = 0$, and $x^2 - ax + c = 0$; find the reducing equation in (b) or (c): and show that it may be depressed to a cubic.

109. If a be a root of Des Cartes's reducing cubic, the four roots of the equation $x^4 + qx^2 + rx + s = 0$,

$$\text{are } -\frac{\sqrt{a}}{2} \pm \sqrt{\left(-\frac{q}{2} - \frac{a}{4} + \frac{r}{2\sqrt{a}}\right)},$$

$$\text{and } +\frac{\sqrt{a}}{2} \pm \sqrt{\left(-\frac{q}{2} - \frac{a}{4} - \frac{r}{2\sqrt{a}}\right)}.$$

110. Prove that Des Cartes's solution of a biquadratic succeeds when all the roots are possible, and two of them equal; and apply it to solve the equation

$$x^4 - 6x^3 + 8x^2 + 6x - 9 = 0.$$

111. Find the roots of the following equations by Des Cartes's method:

1. $x^4 - 4x^3 - 8x + 32 = 0.$
2. $x^4 - 3x^2 - 4x - 3 = 0.$
3. $x^4 - 6x^3 + 5x^2 + 2x - 10 = 0.$
4. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$

112. Give Euler's solution of a biquadratic; and show that the cubic involved in that method has its roots four times less than the roots of the cubic in Des Cartes's.

113. Solve the equation $x^4 = 12x + 5$, by the method attributed to Waring. One root of the reducing cubic is 2.

114. If $a + \beta\sqrt{-1}$ be a root of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

two roots are

$$\tfrac{1}{2}\left\{-(p + 2a) \pm \sqrt{\left[p^2 - 2q - 2\left(a^2 - \beta^2 + \frac{s}{a^2 + \beta^2}\right)\right]}\right\}.$$

115. Prove that if (u) the last term of any equation be resolved into prime factors α, β, γ, so that $u = \alpha^m \beta^n \gamma^p$, then the number of divisors of u will $= (m+1) \cdot (n+1) \cdot (p+1)$.

116. Find by the method of divisors the roots of the following equations:

1. $x^3 - 6x^2 + 5x + 12 = 0.$
2. $x^3 - 9x^2 + 22x - 24 = 0.$
3. $x^4 - 6x^2 - 16x + 21 = 0.$
4. $x^4 - 4x^3 - 8x + 32 = 0.$
5. $x^4 + x^3 - 29x^2 - 9x + 180 = 0.$
6. $3x^3 - 26x^2 + 34x - 12 = 0.$
7. $8x^3 - 26x^2 + 11x + 10 = 0.$
8. $8x^3 - 45x^2 + 73x - 30 = 0.$

117. In the method of divisors, show how the number of substitutions may be lessened:—and in the equation

$$x^4 - x^3 - 16x^2 + 55x - 75 = 0,$$

determine whether 3, 5, and -5 are roots.

118. Apply the rule for quadratic divisors to the equation

$$x^4 - 17x^3 + 88x^2 - 172x + 112 = 0.$$

119. Solve by the method of divisors, the equation

$$6x^4 + 53x^3 - 95x^2 - 25x + 42 = 0.$$

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120. It is always possible to find those roots of numeral equations which are whole numbers or rational fractions without the aid of formulæ of approximation.

121. If a be an approximate root of the equation

$$x^3 + px^2 + qx = r, \text{ so that } a^3 + pa^2 + qa = r_1,$$

prove that $x = a + \dfrac{a \cdot (r - r_1)}{r \text{ or } r_1 + 2a^3 + pa^2}$ very nearly;

r or r_1 being used according as a is greater or less than 1. Approximate by this formula to the value of x in the equation

$$x^3 - 2x = 5.$$

122. Three given quantities $(a+z)$, $(a+z)+h$, $(a+z)+h'$, approximations to the root a of an equation, being substituted for the unknown quantity, give results n, $n+\delta$, $n+\delta'$; show that z will be very nearly found from the equation

$$z^2 \cdot (h\delta' - h'\delta) + z \cdot (h^2\delta' - h'^2\delta) + nhh' \cdot (h' - h) = 0.$$

123. Approximate to a root of the following equations:

1. $x^3 - x - 50 = 0.$
2. $x^3 - 2x - 5 = 0.$
3. $x^3 + 2x - 30 = 0.$
4. $x^3 + x^2 + x = 90.$
5. $x^3 - 6x + 1 = 0.$
6. $x^3 - 2x^2 + 3x - 4 = 0.$
7. $x^4 + x = 3.$
8. $x^4 - 12x + 7 = 0.$
9. $2x^4 - 16x^3 + 40x^2 - 30x + 1 = 0.$
10. $\begin{cases} x^2 + xy = 5, \\ 2xy - y^2 = 2. \end{cases}$
11. $\begin{cases} x^2 + y = 157, \\ y^2 - x = 6. \end{cases}$
12. $\begin{cases} x^3 + y^2 = 12, \\ x^2 + y^3 = 8. \end{cases}$

124. In the equation $x^3 + 9x^2 + 4x = 80$, approximate to the value of x by means of a series of converging fractions.

125. Express the roots of $x^3 - 7x + 7 = 0$, by continued fractions; and determine the accuracy of the approximation of any converging fraction deduced from these.

126. If a be an approximate value of x in any equation, and b, c be the results when a is substituted for x in the original and in the limiting equation; then will

$$x = a - \frac{b}{c}, \text{ nearly.}$$

127. Determine the number of positive and negative roots in the equation $x^5 + 4x^4 - 19x^3 - 34x^2 + 60x + 36 = 0$, of which all the roots are real.

128. Determine the same in the equation
$$x^5 - 5x^4 - 15x^3 + 85x^2 - 26x - 120 = 0.$$

129. Prove without resolving the equation into factors, that if two numbers (a) and (b) when substituted for the unknown quantity in the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, give results affected with contrary signs, there is at least one real root between (a) and (b).

130. If P represent the sum of the positive terms in any equation, and a series of quantities be successively substituted for the unknown quantity; prove that the successive increments of P may be made less than any assignable quantity.

131. If P and N be the greatest positive and negative coefficients in the equation
$$x^n - px^{n-1} \ldots\ldots + Px^{n-m} \ldots\ldots - Nx^{n-r} \ldots\ldots \pm u = 0,$$
then a superior limit to the roots is $N + 1$; and an inferior limit is $\dfrac{u}{u+N}$ or $\dfrac{u}{u+P}$ according as u is positive or negative.

132. If Mx^{n-m} be the first negative term of the equation
$$x^n + px^{n-1} \ldots - Mx^{n-m} - \&c. = 0,$$
and if P be the greatest negative coefficient, then $1 + \sqrt[m]{P}$ is greater than the greatest root of the equation.

133. If $x^n + Ax^{n-1} \ldots - Px^p \ldots - Sx^s \ldots + Tx + V = 0$, where P is the greatest and S the last negative coefficient, then $\dfrac{V^{\frac{1}{s}}}{V^{\frac{1}{s}} + P^{\frac{1}{s}}}$ is an inferior limit of the positive roots.

134. If the terms of an equation be so arranged as to be alternately positive and negative; and if A, A', A'' be the co-

efficients of the 1st, 3rd, 5th.... B, B', B'' of the 2nd, 4th, 6th terms so arranged; show that the greatest of the ratios $\frac{B}{A}$, $\frac{B'}{A'}$, $\frac{B''}{A''}$, is greater than the greatest positive root. Apply this to determine a quantity greater than the greatest root of the equation $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$.

135. If in any equation each negative coefficient be divided by the sum of the positive ones which precede it, and the greatest of these fractions be taken, then this fraction so taken increased by unity is greater than the greatest root of the equation.

136. Let the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. = 0, \text{ be } a, b, c, \&c.;$$

and those of the equation

$nx^{n-1} - (n-1).px^{n-2} + (n-2).qx^{n-3} - \&c. = 0$, be α, β, γ, &c.; then if when α, β, γ, &c. are successively substituted in the equation $x^n - px^{n-1} + \&c. = 0$, the results are P, Q, R, &c., and when a, b, c are substituted in the equation $nx^{n-1} - (n-1).px^{n-2} + \&c. = 0$, the results are p, q, r, &c.; then will $P \times Q \times R \times \&c. : p \times q \times r \times \&c. :: 1 : n^n$.

137. Find a limit greater than the greatest root of the following equations:

1. $x^3 - 6x^2 - 25x - 12 = 0$.
2. $x^3 - 12x^2 + 41x - 43 = 0$.
3. $x^4 - 5x^3 + 6x^2 - 7x + 8 = 0$.

138. Find superior and inferior limits to the positive roots of the equation $x^5 + 2x^4 - 50x^3 - 100x^2 + 49x + 98 = 0$.

139. Determine a limit less than the least positive root of the equation $x^4 - 3x^3 - 5x^2 + 2x + 3 = 0$.

140. Find a limit less than the least root of the following equations:

1. $x^2 + 12x - 20 = 0$.
2. $x^3 + 8x^2 - 8x - 64 = 0$.
3. $x^4 - 5x^2 - 3 = 0$.

141. Find a number greater than the greatest positive root, and also one less than the least negative root of the equation $x^3 - 4x^2 - x + 20 = 0$.

142. Find a limit less than the least difference of the roots of the equation $x^3 - 7x + 7 = 0$.

143. Find between which of the roots of the equation $x^3 - 7x^2 + 7x + 10 = 0$, the number 3 lies.

144. Prove that one of the roots of the equation

$$x^3 - qx - r = 0,$$

when squared, will lie between (q) and $\left(\frac{q}{3}\right)$.

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145. Investigate by means of the limiting equation, whether $x^5 - 5x^3 + 3 = 0$, has a possible root.

146. Determine the rational roots of the equation

$$2x^4 + x^3 - 10x^2 - 2x + 12 = 0.$$

147. Determine the number of possible roots in the equation $x^7 - 14x^4 + 90 = 0$.

148. Determine the number of imaginary roots in the equation $x^5 - 6x^4 + 5x^3 - 30x^2 + 6x - 36 = 0$.

149. Apply Newton's rule for detecting impossible roots, to the equation $x^6 + 3x^4 - 4x^2 - 12 = 0$.

150. The number of impossible roots of any equation

$$x^n - px^{n-1} + \&c. = 0,$$

is not increased by multiplying its terms by the successive terms of the series 0, 1, 2, 3, 4, &c.

151. If several consecutive terms of an equation, whose roots are real, be wanting, and if the next terms on each side of those wanting have the same sign, prove that the equation cannot have as many roots as it has dimensions.

152. Let α, β, γ, &c. be the roots of the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, (m) of which are possible; show that if the equation be transformed into one, whose roots are $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, $(\beta - \gamma)^2$, &c., the last term of the transformed equation will be positive or negative according as $m \cdot \frac{m-1}{2}$ is an even or an odd number.

THE END.

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